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Jacobson Radical and On A Condition for Commutativity of Rings

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Abstract: *Some rings have properties that differ radically from usual number theoretic problems. This fact forces to define what is called Radical of a ring. In Radical theory ideas of Homomorphism and the concept of Semi-simple ring is required where Zorn's Lemma and also ideas of axiom of choice is very important. Jacobson radical of a ring R consists of those elements in R which annihilates all simple right R-module. Radical properties based on the notion of nilpotence do not seem to yield fruitful results for rings without chain condition. It was not until Perlis introduced the notion of quasi-regularity and Jacobson used it in 1945, that significant chainless results were obtained.*
Keywords: *Commutativity, Ideal, Jacobson Radical, Simple ring, Quasi- regular.*

I. Introduction

Firstly, we have described some relevant definitions and Jacobson Radical, Left and Right Jacobson Radical, impact of ideas of Right quasi-regularity from Jacobson Radical etc have been explained with careful attention. Again using the definitions of Right primitive or Left primitive ideals one can find the connection of Jacobson Radical with these concepts. One important property of Jacobson Radical is that any ring R can be embedded in a ring S with unity such that Jacobson Radical of both R and S are same. Another important result is that any nontrivial ring R is Jacobson semi-simple if and only if R has been highlighted with proof.

Then we have discussed to condition for Commutativity of Rings. A well known theorem of Jacobson asserts that if R is a ring such that every element of R is equal to some power of itself then R is commutative. In his proof he used axioms of choice. Herstein gave another proof of the same theorem without involving axiom of choice. Considering the application of this theorem more elementary proof has been given.

1.1. Commutative ring.

Although ring addition is commutative, so that for every $a, b \in R, a + b = b + a$ ring multiplication is not required to be commutative; $a - b$ need not equal $b - a$ for all $a, b \in R$. Rings that also satisfy commutativity for multiplication are called commutative ring.

Formally, let (R, +,-) be a ring. Then (R, +,-) is said to be a commutative ring if for every $a, b \in R, a - b = b - a$. That is, (R, +,-) is required to be a commutative monoid under multiplication.
Example: The integers form a commutative ring under the natural operations of addition and multiplication.

1.2. Non commutative ring.

Non commutative ring is a ring whose multiplication is not commutative; that is $a, b \in R, a - b \neq b - a$ Example: Let $M = \left\{\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} : a \text{ and } b \text{ are real numbers}\right\}$.Then, ($M$, +,-) is a non-commutative ring, without unity and without zero divisors.

1.3. Ring with unity.

A ring R is said to be a ring with unity if it contains an element denoted by 1_R or simply 1 such that $a.1 = 1.a \ \forall a \in R$. The unity element 1 is called multiplicative identity.

Example: The set of all integers **Z** is a commutative ring with unity.

1.4. Division ring.

A ring R in which the set R^* of non-zero elements is a group with respect to the multiplication in R is called a division ring. Equivalently, R is a division ring if every non-zero element of R has a multiplicative inverse in R.
Example: The rings Q, R, C are some examples of division rings.

1.5. Nil and nilpotent.

An element x is said to be nilpotent if there exists a positive integer n such that $x^n = 0$. A ring R is said to nil if every element $x \in R$ is nilpotent, $x^n = 0$, where n depends on the particular element x of R.

II. Jacobson Radical

In Mathematics, more specifically ring theory, a branch of abstract algebra, the Jacobson radical of a ring R consists of those elements in R which annihilate all simple right R –module. The Jacobson radical of a ring R with 1 is defined as the radical ideal of R with respect to the property that "A 2-sided ideal I is such that $1 - a$ is a unit in R for all $a \in I$" and it is denoted by $J(R)$. In other words, $J(R)$ is the largest 2-sided ideal of R such that $1 - a$ is a unit for all $a \in J(R)$.

A computationally convenient notion when working with the Jacobson radical of a ring is the notion of quasi-regularity. In particular, every element of ring's Jacobson radical can be characterized as the unique right ideal of a ring, maximal with respect to the property that each element is right quasi-regular.

The Jacobson radical of a ring is also useful in studying modulus over the ring. For instance, if U is a right R-module and V is a maximal submodule of $U.J(R)$ is contained in V, where $U.J(R)$ denotes all products of elements of $J(R)$ with elements in U. The Jacobson radical may also be defined for rings without unity.

2.1 Left Jacobson radical.

For any ring R with 1, the intersection of all maximal left ideals of R is called the left Jacobson radical or simply the left radical of R and is denoted by $J_l(R)$ (In case R is commutative, $J_l(R)$ is the intersection of all maximal ideals of R.)

Some examples, the left radical of a division ring is (0). More generally, the left radical of $M_n(D)$ is $(0)(\forall n \in N)$ where D is a division ring. The (left) radical of **Z** is (0). The (left) radical of a local ring is its unique maximal ideal.

2.2. Right Jacobson radical.

For any ring R with 1, the intersection of all maximal right ideals of R is called the right Jacobson radical or simply the right radical of R and is denoted by $J_r(R)$.

Note, (In case R is commutative, $J_r(R)$ is the intersection of all maximal ideal of R.)

Some examples, **a)** $J_r(R)$ is a 2-sided ideal of R. b) $J_r(R) = \{x \in R \mid 1 - xy \text{ a unit}, \forall y \in R\}$

2.3. Maximal left ideal.

A left ideal I in R is said to be a maximal left ideal in R if $I \neq R$ and for a left ideal J of R, $I \subseteq J \subseteq R \Rightarrow J = I$ or, $J = R$, i.e., there are no left ideals strictly in between I and R.

2.4. Minimal left ideal.

A left ideal I in R is said to be a minimal left ideal in R if $I \neq 0$ and for a left ideal I of R, $(0) \subseteq J \subseteq I \Rightarrow J = (0)$ or, $J = I$ i.e., there are no left ideals strictly in between (0) and I.

Remark, Maximal (resp. minimal) right/2-sided ideals are defined in exactly the same way as above.

2.5. Right quasi-regular ring.

A ring R is right quasi-regular if every element in it is right quasi-regular.

2.5.1. Right quasi-regular right ideal.

If I is a right ideal of a ring R and if every element of I is right quasi-regular, then I is a right quasi-regular right ideal (or right quasi-regular left ideal or two sided ideal).

2.6. Right primitive.

R is right primitive if R contains a maximal right ideal M such that $(M:R) = 0$

2.6.1. Right primitive ideal.

An ideal (two-sided) P of R is a right primitive ideal if R/P is right primitive.

III. Wedderburn's theorem

If A is a simple ring unit 1 and minimal left ideal I, then A is isomorphic to the ring of $n \times n$ matrices over a division ring.

Proof. Let D be a division ring and $M(n, D)$ be the ring of matrices with entries in D. It is not hard to show that every left ideal in $M(n, D)$ takes the form

$\{M \in M(n,D) \mid \text{The } n_{1,\ldots,}n_k - \text{th columns of } M \text{have zero entires}\}$, for some fixed $\{n_{1,\ldots,}n_k\} \subset \{1,\ldots,n\}$. So a minimal ideal in $M(n,D)$ is of the form $\{M \in M(n,D) \mid \text{All but the k} - \text{th columns have zero entries}\}$, for a given k. In other words, if $I = (M(n,D))\ e$ where e is the idempotent matrix with 1 in the(k,k) entry and zero elsewhere. Also,D is isomorphic to $e\big(M(n,D)\big)e$. The left ideal I is a minimal left ideal, then $= \big(M(n,D)\big)\ e\ \ e\ (k,k)D\ \ e\big(M(n,D)\big)e$. I over $e\big(M(n,D)\big)e$, and the ring $M(n,D)$ is clearly isomorphic to the algebra of homomorphisms on this module. The above example suggests the following lemma.

3.1.Lemma.

A is a ring with identity 1 and an idempotent element e where $AeA = A$. Let I be the left ideal Ae, considered as a right module over eAe. Then A is isomorphic to the algebra of homomorphisms on I, denoted by $Hom(I)$.

Proof. We define the left regular representation $\phi: A \to Hom(I)$ by $\phi(a)m = am$ for $m \in I$. ϕ is injective because if $a.I = aAe = 0$, then $aA = aAeA = 0$, which implies $a = a.1 = 0$. For surjectivity, let $T \in Hom(I)$. Since $AeA = A$, the unit 1 can be expressed as $1 = \sum a_i e b_i$. So $T(m) = T(1.m) = T(\sum a_i e b_i\, m) = T(\sum a_i e e b_i m) = \sum T(a_i e) e b_i m = [\,\sum T(a_i e) e b_i]m$. Since the expression$[\sum T(a_i e) e b_i]$ does not depend on m, ϕ is surjective. This proves the lemma. Wedderburn's theorem follows readily from the lemma3.1.

IV. Zorn's Lemma

A partially ordered non-empty set in which every chain is bounded above (resp. below) has a maximal (resp.minimal) element.

Proof. A non-empty set X with a partial order $' \le '$ is called a poset, i.e. $' \le '$, satisfies the following .
Reflexive. $x \le x, \forall x \in X$,
Anti-symmetry. $x \le y$ and $y \le x \Rightarrow x = y$,
Transitivity. $x \le y$ and $y \le z \Rightarrow x \le z$.
A subject Y of X is called a chain or totally ordered if any two elements of Y are comparable, i.e., given $x, y \in Y$ either $x \le y$ and $y \le x$. In particular, given finitely many elements $y_1, \ldots\ldots\ldots\ldots, y_n$ in Y, there is a permutation σ of $1,2, \ldots\ldots n$ such that $y_{\sigma(1)} \le\ \ldots\ldots.. \le\ y_{\sigma(n)}$.
A subset A of X is said to be bounded above (resp. below) if there is an $\alpha \in X$ such that $a \in \alpha\ (; \alpha \epsilon\ a)\ \forall a \in A$. Such an α is called an upper (resp. a lower) bound for A. It need not belong to A. A subset A of X is said to have a maximal (resp. minimal) element if there is an $a \in A$ such that $\nleq x\ (; x\ \nleq\ a)\ \forall\ a \in A,\ x \ne a$. Note that a maximal (resp. minimal) element need not exists or it need not be an upper (resp. lower) bound when exists or it need not be unique.

5.1. Lemma. (Andrunakievic).

Let A be an ideal of a ring R, B be an ideal of A. Also let B^* be the ideal of R generated by B. Then $B^{*3} \subseteq B$.
Proof. $B^{*3} \subseteq AB^*A = A(B + RB + BR + RBR)A \subseteq ABA \subseteq B$

5.2. Lemma.

If J is the Jacobson radical of a ring R and T is an ideal of R, then thinking of T as a ring, the Jacobson radical of $T = J \cap T$. In particular, if R is semi-simple, then so is T.

Proof. Let A be the Jacobson radical of T and A^* be the ideal of R generated by A. By Lemma 5.1, $A^{*3} \subseteq A$. Now A^* is an ideal of R and since it is in , it is right quasi-regular.
If R is semi-simple, then $A^{*3} = 0, A^*$ is a nilpotent ideal of R, $A^* = 0$, therefore $A = 0$, T is semi-simple. If R is not semi-simple, then $(T + J)/J$ is an ideal of the semi simple ring R/J. Thus, $(T + J)/J$ is semi-simple.
But $\frac{T+J}{J} \cong T(T \cap J)$. Thus $(T + J)/J$ is semi-simple. Now, $T \cap J$ is an ideal of T and it is right quasi-regular. Therefore, $T \cap J \subseteq A$. If $\cap J \ne A$, then $A/(T \cap J)$ is a non-zero right quasi-regular ideal of $T/(T \cap J)$. This is impossible. Therefore, $T \cap J = A$.
Remark. Lemma5.2 tells us that every ideal of a Jacobson radical ring is itself Jacobson Radical.

5.3. Lemma.

Any ring R can be embedded in a ring S with unity such that, Jacobson radical of R = Jacobson radical of S.

Proof. If R has a unity, then we merely take $S = R$. If R does not have a unity, then we adjoin one in the standard way by taking ordered pairs of elements of R and integers. Call this large ring S. Then R is an ideal of S and $S/R \cong$ integers. Since S/R is semi-simple, R must contains J_S, the Jacobson radical of S , on the other hand, by lemma 5.2, J_R the Jacobson radical of R, is equal to $J_S \cap R$. Since $J_S \subseteq R$, we have $J_S = J_R$.

5.4. Lemma.
A simple non-trivial ring R is Jacobson semi-simple if and only if R has some maximal right ideals.

Proof. If R is simple, non-trivial (i.e.$R^2 \neq 0$), and not Jacobson radical, then there exists an element x in R which is not right quasi-regular. Then, R contains maximal right ideals.
Conversely, if R is simple, non trivial, and contains a maximal right ideal M, take x in R, x not in M. Consider xR Since M is maximal, xR cannot be in M. To see this, consider $\{y: yR \subseteq M\}$.
This set is clearly a right ideal of R, and it certainly contains M. However, it cannot be all of R because RR is a non-zero ideal of R and must be equal to because R is simple. Thus $RR \nsubseteq M$. Consequently, $\{y: yR \subseteq M\}$. must be equal to M. Then if $\subseteq M$, x is in M.
Since we selected x not in M, xR is not in M. Then R equals the right ideal generated by M and xR. In particular, there must exist m in M and x' in R, such that $x = m + xx'$. Let $R_x = (\{a: xa\epsilon M\}$. This is a right ideal. We observe that $\{c - x'c\}$ is contained in R_x for $x(c - x'c) = (x - xx') = mc$ is in M. Now if R is Jacobson radical, then $-x'$ is right quasi-regular and there exist an element z. Such that $-x' + z - x'z = 0$. Then $x' = z - x'z = 0$ and this in R_x. Therefore, $x'c \in R_x$ for every c. Then c is in R_x for every c ,$R_x = R$. However, in that case $xR \subseteq M$, a contradiction. Therefore, $-x'$ cannot be right quasi-regular, R is not Jacobson radical and therefore, R is Jacobson semi-simple.

Remark. If a simple ring is trivial, then it is nilpotent and therefore Jacobson Radical.

V. On a condition for Commutivity of Rings.

6.1. Jacobson Theorem.
Let R be a ring such that every element in R is a power of itself. Then R is commutative.
Proof**.**
To proves this theorem, we first prove the three important lemmas.

6.1.1. Lemma.
Let R be a non-commutative ring such that $x \in R$, there exists a positive integer $n(x) > 1$, such that $x^{n(x)} = x$. Then R contains a non-commutative subring of prime characteristic.
Proof. By non-commutative, there exists an x in R, such that x is not in the centre of R , so it will sufficient to show that $x, y \in R$ such that $px = py = 0$, where p is a prime, and $xy \neq yx$. Suppose that $x^n = x, (2x)^m = 2x,$ then $S = (n-1)(m-1) + 1$, we have $(2x)^s = 2^s x^s = 2^s x = 2x$, whence $(2^s - 2)x = 0$.
Suppose that $tx = 0$ with $|t|$ minimal, then since R contains no nilpotent, t is square-free; whence x is the sum of elements each of prime characteristic. Clearly, $px = py = 0$, with p and q coprime implies that $xy = yx = 0$; whence the result follows.

6.1.2. Lemma.
Let R be a ring of prime characteristic satisfying the conditions of lemma 6.1.1. Then R contains a finite non-commutative subring.

Proof. Each x belonging to R generates a subring which is the direct product of finite fields. Take some finite field F, of order P^S, generated by an x which is not in the centre of R . If R is non-commutative and x is not in the centre of R then $x \in x^{n(x)-1}\, R\, x^{n(x)-1}$
and, since

$$R = x^{n(x)-1}\, R\, x^{n(x)-1} \oplus \left(x^{n(x)-1} - 1\right) R (x^{n(x)-1} - 1) ,$$

x is not in the centre of $x^{n(x)-1}\, R\, x^{n(x)-1}$; whence, by considering $x^{n(x)-1}\, R\, x^{n(x)-1}$, we may assume that the identity for F is the identity for R.
Letting y be an element of R which does not commute with x, we have

$$y\left(T_x^{P^s} - T_x\right) = y(T_x - I\lambda_1)(T_x - I\lambda_2) \dots \dots (T_x - I\lambda_{P^s}) = 0,$$

Where $\lambda_1, \dots \dots \lambda_{P^s}$, are the elements of F , T_x is the mapping $R \to R$ defined by $(r)T_x = rx - xr$ and $I\lambda_i$ with $\lambda_i \in F$ is defined by $(r)I\lambda_i = \lambda_i r$. With $\lambda_1 = 0$, we have an i such that

$$z = y(T_x - \lambda_1) \dots \dots \dots \dots (T_x - I\lambda_{i-1}) \neq 0,$$

and

$$z(T_x - I\lambda_i) = 0,\ \lambda_i \neq 0;$$

Whence the subring generated by x and z is finite.

6.1.3.Lemma.

Let R be a finite non-commutative ring such that every element of R is a power of itself such that every element of R is a power of itself. Then R contains a non-commutative division ring.

Proof.

Assume that S is a minimal non-commutative subring of R. Then R, itself considered as a ring, contains no zero divisor; for if $a, b \in S$ with $ab = 0, a \neq 0, b \neq 0$ **.**

We have, $S = a^{n(a)-1} S a^{n(a)-1} \oplus \left(a^{n(a)-1} - 1\right) S \left(a^{n(a)-1} - 1\right)$,

With $a \in a^{n(a)-1} S a^{n(a)-1}$ and $b \in \left(a^{n(a)-1} - 1\right) S \left(a^{n(a)-1} - 1\right)$, which contradicts the minimality of S, since the commutativity of both $a^{n(a)-1} S a^{n(a)-1}$ and $\left(a^{n(a)-1} - 1\right) S \left(a^{n(a)-1} - 1\right)$ implies that S is commutative.

Hence lemma 6.1.1, 6.1.2, and 6.1.3 together with Wedderburn's theorem prove the Jacobson theorem.

VI. Conclusion

Looking the problem in difficult way lemma6.1.1 depicts that if R be a non-commutative ring such that for every x belonging in R there exists a positive integer $n(x) \geq 1$ such that $x^{n(x)} = x$ then R contains a non-commutative subring of prime characteristic.

Lemma 6.1.2 ensures that if R be a ring of prime characteristic satisfying the conditions of lemma 6.1.1, then R contains a finite non-commutative subring.

Lemma 6.1.3 also ensures that if R be a finite non-commutative ring such that every element of R is a power of itself then R contains a non-commutative division ring .

Lemma 6.1.1, 6.1.2 and 6.1.3 together with Wedderburn's theorem establishes that, if R be a ring with each element being power of itself then R must be commutative.

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