Total Coloring of Some Cycle Related Graphs

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Abstract: A total coloring of a graph \( G \) is a proper coloring with additional property that no two adjacent or incident graph elements receive the same color. The total chromatic number of a graph \( G \) is the smallest positive integer for which \( G \) admits a total coloring. Here, we investigate the total chromatic number of some cycle related graphs.

Keywords: Middle graph, One point union of cycles, Shadow graph, Total coloring, Total chromatic number, Total graph.

I. Introduction

We begin with finite, connected and undirected graph \( G \), without loops and parallel edges, with vertex set \( V(G) \) and edge set \( E(G) \). The vertices and edges are commonly addressed as graph elements. For any graph theoretic terminology we refer to Chartrand and Lesniak [1]. A proper \( k \)-coloring of a graph \( G \) is a function \( c : V(G) \to \{1, 2, \ldots, k\} \) such that \( c(u) \neq c(v) \) for all \( uv \in E(G) \). The chromatic number \( \chi(G) \) is the minimum integer \( k \) for which the graph \( G \) admits a proper coloring. Some variants of graph coloring are also introduced. Some of them are \( a \)-coloring, \( b \)-coloring, total coloring etc. The present work is focused on total coloring of graphs.

A function \( \pi : V(G) \cup E(G) \to \mathbb{N} \) is called a total coloring if no two adjacent or incident graph elements are assigned the same color. The total chromatic number of \( G \), denoted by \( \chi_T(G) \), is the smallest positive integer \( k \) for which there exists a total coloring \( \pi : V(G) \cup E(G) \to \{1, 2, \ldots, k\} \). The Total Coloring Conjecture(TCC) was posed independently by Behzad [2] and Vizing [3] which states that,

For any graph \( G \), \( \chi_T(G) \leq \Delta(G) + 2 \).

The TCC is open even after many efforts to settle it. It is proved for particular graph families. For e.g., Rosenfeld [4] and Vijayaditya [5] proved it for graphs \( G \) having \( \Delta(G) \leq 3 \). A survey on total coloring of graphs is given in a paper by Behzad [6]. The TCC for complete graphs and complete multi partite graphs have been proved by Behzad et al. [7] and Yap [8]. The work of Yap [9], Andersen [10], Sanders and Zhao [11] as well as Borodin [12] reveals that the TCC is true for planar graphs \( G \) having \( \Delta(G) \neq 5 \). The concept of total coloring is further explored by Xie and Yang [13], Wang [14] and Wang et al. [15].

In the present work we investigate the total chromatic number for the graphs obtained from cycle by means of various graph operations.

Conjecture 1.1 [2] \( \Delta(G) + 1 \leq \chi_T(G) \leq \Delta(G) + 2 \).

Proposition 1.2 [2] A graph \( G \) is said to be of type I if \( \chi_T(G) = \Delta(G) + 1 \) and is of type II if \( \chi_T(G) = \Delta(G) + 2 \).

Proposition 1.3 [16] Any 4- regular multigraph can be total colored with six colors

Proposition 1.4 [17] A cycle of length congruent to \( 0 (\bmod 3) \) is of type I graph and all other cycles are of type II graphs.

II. Main Results

Definition 2.1 A middle graph \( M(G) \) of a graph \( G \) is the graph whose vertex set is \( V(G) \cup E(G) \) and in which two vertices are adjacent whenever either they are adjacent edges of \( G \) or one is vertex of \( G \) and other is an edge incident with it.

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Theorem 2.2 $\chi_t (M(C_n)) = 5$, for all $n$.

Proof: Let $V(C_n) = \{v_1, v_2, ..., v_n\}$ and $E(C_n) = \{e_1, e_2, ..., e_n\}$. Thus, $V(M(C_n)) = \{v_1, v_2, ..., v_n, e_1, e_2, ..., e_n\}$. We observe that in $M(C_n)$, the vertices $v_1, v_2, ..., v_n$ form a cycle of length $n$.

When $n \equiv 0 (\mod 3)$, the colors 1, 2 and 3 can be assigned on vertices and edges successively for the total coloring. For the remaining edges incident on each $e_i$, we must use two new colors, say $\pi = 1, \pi = 5$, successively. The colors 3, 2 and 1 successively can be assigned on the vertices $v_1, v_2, ..., v_n$. Thus only 5 colors suffice for the total coloring.

When $n \not\equiv 0 (\mod 3)$, we are considering following two cases:

Case 1: If the cycle formed by the vertices $e_1, e_2, ..., e_n$ has length $2k; k \geq 2$.

The colors 1 and 2 can be used on consecutive vertices and the colors 3 and 4 on edges of the cycle. For the remaining edges incident on $e_i$ with color 1 we can use the colors 2 and 5, and the edges incident on $e_i$ with color 2 we can use the colors 1 and 5. Thus only 5 colors suffice to color all the elements of $M(C_n)$.

Case 2: If the cycle formed by the vertices $e_1, e_2, ..., e_n$ has length $2k + 1; k \geq 2$.

We can assign the colors as $\pi(e_i) = 1$ for odd $i$, $\pi(e_i) = 2$ for even $i$ and $\pi(e_{2k+i}) = 3, \pi(e_{i+1}) = 3$ for odd $i$, $\pi(e_{i+1}) = 4$ for even $i$. $\pi(v_i e_i) = 4$, $\pi(v_i e_i) = 2$, $\pi(v_1 e_i) = 4$, $\pi(v_1 e_i) = 2$ for odd $i \neq 1$, $\pi(v_1 e_i) = 1$ for even $i$, $\pi(v_1 e_i) = 5$, $\pi(v_1 e_i) = 1, \pi(e_{2k+i} v_{2k+i}) = 5$. Thus only 5 colors suffice to color all the elements of $M(C_n)$. Hence $\chi_t (M(C_n)) = 5$ for all $n$.

Definition 2.3 A total graph $T(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in $G$.

Theorem 2.4 $\chi_t (T(C_n)) = \begin{cases} 5, & n = 3 \\ 6, & n \neq 3. \end{cases}$

Proof: Let $V(C_n) = \{v_1, v_2, ..., v_n\}$ and $E(C_n) = \{e_1, e_2, ..., e_n\}$. Thus, $V(T(C_n)) = \{v_1, v_2, ..., v_n, e_1, e_2, ..., e_n\} = V(M(C_n))$ and $E(T(C_n)) = E(M(C_n)) \cup \{v_1 v_i ; i = 1, 2, ..., n-1\} \cup \{v_i v_n\}$.

As $T(C_n)$ is a regular graph with $\Delta = 4$, $\chi_t (T(C_n)) \leq \Delta(G) + 2 = 6$ by Proposition 1.3.

By the definition of $T(C_n)$, $M(C_n) \subset T(C_n)$, $\chi_t (T(C_n)) \geq \chi_t (M(C_n)) = 5$.

When $n = 3$, assign the colors as in $M(C_n)$ and for the edges $v_i v_{i+1}$ and $v_1 v_n$, we can use the colors which is same as the colors used for $e_i$; $i = 1, 2$ and 3 respectively. Thus $\chi_t (T(C_3)) = 5$.

When $n > 3$, as each vertex is adjacent to exactly four vertices of same order and due to the adjacency and incidence of elements, five colors will not suffice for the total coloring. Thus $\chi_t (T(C_n)) \neq 5$. Hence $\chi_t (T(C_n)) = 6$.

Definition 2.5 The Shadow graph $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$, say $G'$ and $G''$. Join each vertex $u'$ in $G'$ to the neighbors of the corresponding vertex $u''$ in $G''$.

Theorem 2.6 $\chi_t (D_2(C_n)) = \chi_t (C_n) + 2$. 

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Proof: $D_2(C_n)$, the shadow graph of $C_n$ is constructed by taking two copies of $C_n$, say $C'_n$ and $C''_n$. Join each vertex $u'$ in $C'_n$ to the neighbors of the corresponding vertex $u''$ in $C''_n$. It is clear that $v'_i$ and $v''_i$ are non adjacent. So we can assign the same colors for the elements of both $C'_n$ and $C''_n$.

Also $\chi_x(C_n)=3$ when $n \equiv 0(\mod 3)$ and 4 otherwise. Now, $D_2(C_n)$ is a regular graph with degree four and the edges $v'_i, v''_i$ are non adjacent to each other. Only two new colors are required for the coloring of these edges. Thus $\chi_x(D_2(C_n))=\chi_x(C_n)+2$.

Definition 2.7 The one point union $C_n^{(k)}$ of $k$-copies of cycle $C_n$ is the graph obtained by taking $v$ as a common vertex such that any two distinct cycles $C_n^{(i)}$ and $C_n^{(j)}$ are edge disjoint and do not have any vertex in common except $v$.

Theorem 2.8 $\chi_x(C_n^{(k)})=2k+1, \quad k \geq 2$, for all $n$.

Proof: Consider the one point union $C_n^{(k)}$ of $k$-copies of cycle $C_n$ with the common vertex $v$. By the construction of the graph $d(v)=2k$, so we need minimum $2k+1$ colors for the total coloring of the vertex $v$ and the edges incident on it. As the remaining vertices are adjacent to maximum two vertices, we need only $2k+1$ colors for the total coloring of the graph. Thus, $\chi_x(C_n^{(k)})=2k+1$, for all $n$ and $k \geq 2$.

III. Concluding Remarks

The total chromatic number of $C_n$ was investigated by Rosenfeld [4]. But we have explored the concept of total coloring for the larger graphs obtained from $C_n$. We have investigated the total chromatic numbers for middle graph, total graph, shadow graph of cycle as well as one point union of cycles.

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References


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