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# Total Coloring of Some Cycle Related Graphs

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***Abstract:*** *A total coloring of a graph G is a proper coloring with additional property that no two adjacent or incident graph elements receive the same color. The total chromatic number of a graph G is the smallest positive integer for which G admits a total coloring. Here, we investigate the total chromatic number of some cycle related graphs.*
***Keywords:*** *Middle graph, One point union of cycles, Shadow graph, Total coloring, Total chromatic number, Total graph.*

## I. Introduction

We begin with finite, connected and undirected graph $G$, without loops and parallel edges, with vertex set $V(G)$ and edge set $E(G)$. The vertices and edges are commonly addressed as graph elements. For any graph theoretic terminology we refer to Chartrand and Lesniak [1]. A proper $k$-coloring of a graph $G$ is a function $c: V(G) \rightarrow \{1, 2, ..., k\}$ such that $c(u) \neq c(v)$, for all $uv \in E(G)$. The chromatic number $\chi(G)$ is the minimum integer $k$ for which the graph $G$ admits a proper coloring. Some variants of graph coloring are also introduced. Some of them are $a$ - coloring, $b$ - coloring, total coloring etc. The present work is focused on total coloring of graphs.

A function $\pi: V(G) \cup E(G) \rightarrow \mathbf{N}$ is called a *total coloring* if no two adjacent or incident graph elements are assigned the same color. The total chromatic number of $G$, denoted by $\chi_T(G)$, is the smallest positive integer $k$ for which there exists a total coloring $\pi: V(G) \cup E(G) \rightarrow \{1, 2, ..., k\}$. The Total Coloring Conjecture(TCC) was posed independently by Behzad [2] and Vizing [3] which states that,

*For any graph* $G$, $\chi_T(G) \leq \Delta(G) + 2$.

The TCC is open even after many efforts to settle it. It is proved for particular graph families. For e.g., Rosenfeld [4] and Vijayaditya [5] proved it for graphs $G$ having $\Delta(G) \leq 3$. A survey on total coloring of graphs is given in a paper by Behzad [6] . The TCC for complete graphs and complete multi partite graphs have been proved by Behzad *et al*. [7] and Yap [8]. The work of Yap [9], Andersen [10], Sanders and Zhao [11] as well as Borodin [12] reveals that the TCC is true for planar graphs $G$ having $\Delta(G) \neq 5$. The concept of total coloring is further explored by Xie and Yang [13], Wang [14] and Wang *et al* . [15].

In the present work we investigate the total chromatic number for the graphs obtained from cycle by means of various graph operations.

**Conjecture 1.1** [2] $\Delta(G) + 1 \leq \chi_T(G) \leq \Delta(G) + 2$.

**Proposition 1.2** [2] A graph $G$ is said to be of type I if $\chi_T(G) = \Delta(G) + 1$ and is of type II if $\chi_T(G) = \Delta(G) + 2$.

**Proposition 1.3** [16] Any 4- regular multigraph can be total colored with six colors

**Proposition 1.4** [17] A cycle of length congruent to $0 \pmod 3$ is of type I graph and all other cycles are of type II graphs.

## II. Main Results

**Definition 2.1** A middle graph $M(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent whenever either they are adjacent edges of $G$ or one is vertex of $G$ and other is an edge incident with it.

**Theorem 2.2** $\chi_T(M(C_n))$=5, for all $n$.

*Proof*: Let $V(C_n)=\{v_1,v_2,...,v_n\}$ and $E(C_n)=\{e_1,e_2,...,e_n\}$. Thus, $V(M(C_n))$= {$v_1,v_2,...,v_n$, $e_1,e_2,...,e_n$}. We observe that in $M(C_n)$, the vertices $e_1,e_2,...,e_n$ forms a cycle of length $n$.

When $n\equiv 0(\mathrm{mod}\,3)$, the colors 1, 2 and 3 can be assigned on vertices and edges successively for the total coloring. For the remaining edges incident on each $e_i$, we must use two new colors, say 4 and 5, successively. The colors 3, 2 and 1 successively can be assigned on the vertices $v_1,v_2,...,v_n$. Thus only 5 colors suffice for the total coloring.

When $n\not\equiv 0(\mathrm{mod}\,3)$, we are considering following two cases:

**Case 1 :** If the cycle formed by the vertices $e_1,e_2,...,e_n$ has length $2k$; $k\geq 2$.

The colors 1 and 2 can be used on consecutive vertices and the colors 3 and 4 on edges of the cycle. For the remaining edges incident on $e_i$ with color 1 we can use the colors 2 and 5, and the edges incident on $e_i$ with color 2 we can use the colors 1 and 5. Thus only 5 colors suffice to color all the elements of $M(C_n)$.

**Case 2 :** If the cycle formed by the vertices $e_1,e_2,...,e_n$ has length $2k+1$; $k\geq 2$.

We can assign the colors as $\pi(e_i)=1$ for odd $i$, $\pi(e_i)=2$ for even $i$ and $\pi(e_{2k+1})=3$, $\pi(e_ie_{i+1})=3$ for odd $i$, $\pi(e_ie_{i+1})=4$ for even $i$, $\pi(e_1e_{2k+1})=2$, $\pi(v_1e_1)=4$, $\pi(v_ie_i)=2$ for odd $i\neq 1$, $\pi(v_ie_i)=1$ for even $i$, $\pi(e_iv_{i+1})=5$, $\pi(e_{2k+1}v_{2k+1})=1$, $\pi(e_{2k+1}v_1)=5$. Thus only 5 colors suffice to color all the elements of $M(C_n)$. Hence $\chi_T(M(C_n))$=5 for all $n$.

**Definition 2.3** A total graph $T(G)$ of a graph $G$ is the graph whose vertex set is $V(G)\cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in $G$.

**Theorem 2.4** $\chi_T(T(C_n))=\begin{cases}5, & n=3\\ 6, & n\neq 3.\end{cases}$

Proof: Let $V(C_n)=\{v_1,v_2,...,v_n\}$ and $E(C_n)=\{e_1,e_2,...,e_n\}$. Thus, $V(T(C_n))$ = {$v_1,v_2,...,v_n$, $e_1,e_2,...,e_n$} = $V(M(C_n))$ and $E(T(C_n))$ = $E(M(C_n))\cup\{v_iv_{i+1};\, i=1,2,...,n-1\}\cup\{v_1v_n\}$.

As $T(C_n)$ is a regular graph with $\Delta$= 4, $\chi_T(T(C_n))\leq\Delta(G)+2$= 6 by Proposition 1.3.

By the definition of $T(C_n)$, $M(C_n)\subset T(C_n)$, $\chi_T(T(C_n))\geq\chi_T(M(C_n))$=5.

When $n$ = 3, assign the colors as in $M(C_n)$ and for the edges $v_1v_2$, $v_2v_3$ and $v_3v_1$, we can use the colors which is same as the colors used for $e_i$; $i$ = 1, 2 and 3 respectively. Thus $\chi_T(T(C_3))$=5.

When $n>3$, as each vertex is adjacent to exactly four vertices of same order and due to the adjacency and incidence of elements, five colors will not suffice for the total coloring. Thus $\chi_T(T(C_n))\neq 5$. Hence $\chi_T(T(C_n))$ =6.

**Definition 2.5** The Shadow graph $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$, say $G'$ and $G''$. Join each vertex $u'$ in $G'$ to the neighbors of the corresponding vertex $u''$ in $G''$.

**Theorem 2.6** $\chi_T(D_2(C_n))$= $\chi_T(C_n)$+2.

**Proof:** $D_2(C_n)$, the shadow graph of $C_n$ is constructed by taking two copies of $C_n$, say $C_n'$ and $C_n''$. Join each vertex $u'$ in $C_n'$ to the neighbors of the corresponding vertex $u''$ in $C_n''$. It is clear that $v_i'$ and $v_i''$ are non adjacent. So we can assign the same colors for the elements of both $C_n'$ and $C_n''$.

Also $\chi_T(C_n)$=3 when $n \equiv 0(\text{mod}\, 3)$ and 4 otherwise. Now, $D_2(C_n)$ is a regular graph with degree four and the edges $v_i'v_i''$ are non adjacent to each other. Only two new colors are required for the coloring of these edges. Thus $\chi_T(D_2(C_n))$ = $\chi_T(C_n)$+2.

**Definition 2.7** The one point union $C_n^{(k)}$ of $k$-copies of cycle $C_n$ is the graph obtained by taking $v$ as a common vertex such that any two distinct cycles $C_n^{(i)}$ and $C_n^{(j)}$ are edge disjoint and do not have any vertex in common except $v$.

**Theorem 2.8** $\chi_T(C_n^{(k)})$ = $2k$+1, $k \geq 2$, for all $n$.

Proof: Consider the one point union $C_n^{(k)}$ of $k$-copies of cycle $C_n$ with the common vertex $v$. By the construction of the graph $d(v)$ =$2k$, so we need minimum $2k$+1 colors for the total coloring of the vertex $v$ and the edges incident on it. As the remaining vertices are adjacent to maximum two vertices, we need only $2k$+1 colors for the total coloring of the graph. Thus, $\chi_T(C_n^{(k)})$ =$2k$+1, for all $n$ and $k \geq 2$.

## III. Concluding Remarks

The total chromatic number of $C_n$ was investigated by Rosenfeld [4]. But we have explored the concept of total coloring for the larger graphs obtained from $C_n$. We have investigated the total chromatic numbers for middle graph, total graph, shadow graph of cycle as well as one point union of cycles.

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