Various Reflexivities in Sequence Spaces

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Abstract: In this paper, we investigate that for each $p, 1 < p < \infty$, space $\ell^p$, equipped with the norm topology, is both (i) B-reflexive, and (ii) inductively reflexive. We also discuss that the locally convex spaces $\ell^p [\tau(u)]$, where $1 < p < \infty$ and $1/p + 1/q = 1$, are semi-reflexive (and so polar semi-reflexive) and the locally convex space $\ell^1 [\tau(c_0)]$ is inductively semi-reflexive.

Keywords: Bornological, B-reflexive, inductively reflexive, normed topology, polar reflexive, sequence space.

I. Introduction

Semi-reflexivity and reflexivity are well known properties in locally convex spaces. There are other types of reflexivity, namely, polar semi-reflexivity and polar reflexivity in [1], inductive semi-reflexivity and inductive reflexivity introduced by I.A. Berezanskij [2] and B-semireflexivity, B-reflexivity in [3]. The notions of $p$-completeness and $p$-reflexivity introduced by Kalman Braun [4] are nothing but polar semi-reflexivity and polar reflexivity, respectively. In this paper we discuss these reflexivity in sequence spaces. For a locally convex space $E[\tau]$, which we always consider Hausdorff, the dual is denoted by $E'$. The strong dual of $E[\tau]$ is $E'[\tau_0(E)]$ and the bidual of $E[\tau]$ is $E''[\tau_0(E)']$. We follow the notion of Köthe [1] for notations and terminology, unless specifically mentioned.

A locally convex space $E[\tau]$ is called semi-reflexive if $E = E''$. A semi-reflexive locally convex space $E[\tau]$ is called reflexive provided $\tau = \tau_0(E')$.

Let $\tau^0$ be the topology on $E'$ of uniform convergence over the class of $\tau$-precompact sets (in E). We have $\tau^0 \subseteq \tau_0(E)$. The topology on $(E'[\tau^0])'$ of uniform convergence over the class of $\tau^0$-precompact subsets of $E'[\tau^0]$ is denoted as $\tau^{\omega 0}$.

1.1. Definition (Köthe [1]): A locally convex space $E[\tau]$ is called polar semi-reflexive if $E = (E'[\tau^0])'$. Polar semi-reflexive space $E[\tau]$ is called polar reflexive if $\tau = \tau^{\omega 0}$, i.e. $(\tau^0)^\alpha$.

Consider a locally convex space $E[\tau]$ and a base $\{U_\alpha: \alpha \in I\}$ of $\tau$-neighborhoods of 0 consisting of closed absolutely convex neighborhoods. Let $U^0_\alpha$ be the polar of $U_\alpha$ in $E'$ and $E'_{U_\alpha}$ be the linear subspace of $E'$ spanned by $U^0_\alpha$ equipped with the norm topology with $U^0_\alpha$ as unit ball. Let $E'[\tau']$ be the inductive limit of the system $\{E'_{U_\alpha}\}$ and the embeddings $E'_{U_\alpha} \rightarrow E'$. Note that $\tau'$ is the finest locally convex topology on $E'$ making all embeddings $E'_{U_\alpha} \rightarrow E'$ continuous. Starting from the locally convex space $E'[\tau']$, the topology $\tau'' = (\tau')'$ is defined on $(E'[\tau'])'$. The topology $\tau''$ constructed this way is due to [2].

1.2. Definition (Berezanskij [2]): If $(E'[\tau])' = E$, then $E[\tau]$ is called inductively semi-reflexive. If, in addition, $\tau = (\tau')'$, then $E[\tau]$ is called inductively reflexive.

Following P.K. Raman [3], we define that an absolutely convex bounded subset $B$ of the dual $E'$ of a l.c. space $E[\tau]$ is called reflective if the span $E_B$ is a reflexive Banach space with $B$ as unit ball. The class of all reflective sets is denoted by $\mathcal{R}$. The topology on $E$ of uniform convergence over the saturated class of sets generated by $\mathcal{R}$ is called the reflective topology of $E$ and is denoted by $\tau$. The polars of the sets of $\mathcal{R}$ i.e. the class $\{K_0: K \in \mathcal{R}\}$ forms a base of neighborhoods of the origin $0$ for $E[\tau]$.

1.3. Definition (Raman [3]): If a locally convex space $E[\tau]$ is barreled and $E^\square = \text{the completion of } E[\tau]$, then $E[\tau]$ is said to be B-semireflexive if $E = E^\square$ algebraically.

If, in addition, $\tau = \tau$, we say that $E[\tau]$ is B-reflexive.

Let us denote

$t^\omega = \text{The set of all bounded sequences } x = \{x_k\} \text{ of real or complex numbers.}$
c = The set of all convergent sequences \( x = \{ \xi_k \} \) of real or complex numbers.

c_0 = The set of all sequences \( x = \{ \xi_k \} \) of real or complex numbers which are convergent to 0.

t_1 = The set of all sequences \( x = \{ \xi_k \} \) of real or complex numbers with \( \sum_{k=1}^{\infty} |\xi_k| < \infty \).

\( \ell^p, 1 < p < \infty \) = The set of all sequences \( x = \{ \xi_k \} \) of real or complex numbers for which \( \sum_{k=1}^{\infty} |\xi_k|^p \) converges.

If a bounded sequence \( x = \{ \xi_k \} \) is considered as a coordinate vector \( x = (\xi_k) \), then the coordinate-wise addition and scalar multiplication i.e. for all \( x = \{ \xi_k \}, y = \{ \eta_k \} \in \ell^\infty \) and \( \alpha \in \mathbb{K} \), \( x+y = \{ \xi_k + \eta_k \} \) and \( \alpha x = \{ \alpha \xi_k \} \), define a vector space structure on \( \ell^\infty \) (and on \( c, c_0, t_1 = \ell^p, 1 \leq p < \infty \), as well). Such vector spaces are known as sequence spaces. In subspace relationship, we have \( t_1 \subseteq c_0 \subseteq c \subseteq \ell^\infty \). The (usual) norm on \( \ell^\infty \) is \( \| x \| = \sup \{ |\xi_k| : k \geq 1 \} \). On \( \ell^p, 1 \leq p < \infty \), the norm \( \| x \|_p \) is given by is \( \| x \|_p = (\sum_{k=1}^{\infty} |\xi_k|^p)^{1/p} \). In particular, on \( t^1 \), the norm \( \| x \|_1 \) is given by is \( \| x \|_1 = \sum_{k=1}^{\infty} |\xi_k| \) and on \( t^2 \), the norm \( \| x \|_2 \) is given by is \( \| x \|_2 = (\sum_{k=1}^{\infty} |\xi_k|^2)^{1/2} \).

1.4. Following facts are well known:

(i) Each of \( t^\infty, c, \) and \( c_0 \), equipped with the norm \( \| x \|_\infty \), is a (B)-space.

(ii) \( \ell^p, 1 \leq p < \infty \), with the norm \( \| x \|_p \), are (B)-spaces.

1.5. Further, we have the following dual relationships between \( t^\infty, c, \) and \( c_0 \), \( t^1 \) and \( \ell^p, 1 \leq p < \infty \):

\( (t^1)' = \ell^\infty \); \( (c_0)' = t^1 \); \( (c)' = c \) and for each \( p, 1 < p < \infty \), \( (\ell^p)' = \ell^q \) where, \( 1/p + 1/q = 1 \). For details of these results see [1], §14.7& 8.

1.6. It is observed that \( (c_0)' = (t^1)' = \ell^\infty \) therefore \( c_0 \) is not reflexive. Similarly \( t^1 \) and \( \ell^\infty \) are also non-reflexive. However for each \( p, 1 < p < \infty \), \( (\ell^p)' = \ell^q \), where, \( 1/p + 1/q = 1 \), and so, \( (\ell^p)' = (\ell^p)' = \ell^p \) and therefore each \( \ell^p \) is a reflexive (B)-space.

In this paper, we discuss polar reflexivity, B-reflexivity, Inductive reflexivity on these sequence spaces considered with their normed topologies, and sometimes with weak or Mackey topology.

II. Results

We know that (F)-spaces are always polar reflexive ([1], §23.9(5)). So each of the (B)-spaces \( \ell^\infty, c, \) and \( c_0 \) (equipped with the norm \( \| x \|_\infty \)) and \( \ell^p, 1 \leq p < \infty \) (equipped with the norm \( \| x \|_p \)) is polar reflexive.

Let \( \tau_p \) be the usual normed topology on the (B)-space \( \ell^p, 1 < p < \infty \), with respect to the norm \( \| x \|_p \) given by \( \| x \|_p = (\sum_{k=1}^{\infty} |\xi_k|^p)^{1/p} \). Now we have the following assertion:

2.1. Theorem: For each \( p, 1 < p < \infty \), the locally convex space \( \ell^p[\tau_p] \) is both (i) B-reflexive, and (ii) inductively reflexive.

Proof: (i) It is already known that \( U[\tau_p] \) is a reflexive (B)-space (see 1.6). So its strong dual \( \ell^q[\tau_q] \), where \( 1/p + 1/q = 1 \), is also a reflexive (B)-space. Thus \( U[\tau_p] \) is a reflexive and its strong dual is bornological. Hence, by [3], theorem 17, \( \ell^p[\tau_p] \) is B-semireflexive. Further, the fact that \( U[\tau_p] \) is a reflexive (B)-space implies that for the unit ball \( S \) of \( U[\tau_p] \), the polar \( S^* \) in the dual \( t^1 \) is a reflective set. Hence the reflective topology \( \tau_1 \) on \( t^1 \) is finer than the normed topology \( \tau_p \) and consequently we have \( \tau_p \supseteq \tau_1 \). Hence \( \ell^p[\tau_p] \) is B-reflexive.

(ii) Since B-semireflexivity implies inductive semi-reflexivity ([5], theorem 2.4), \( \ell^p[\tau_p] \) is inductively semi-reflexive. Further, \( \ell^p[\tau_p] \) is a (B)-space and so it is bornological. A locally convex space which is inductively semi-reflexive and bornological is inductively reflexive ([2], theorem 1.7). Hence \( \ell^p[\tau_p] \) is inductively reflexive.

In particular, for \( p = 2 \), we have

2.2. Corollary: The (B)-space \( t^2 \) is both B-reflexive and inductively reflexive.
2.3. Theorem: The locally convex space \( \ell^p [t_\alpha(t^p)] \), where \( 1 < p < \infty \) and \( 1/p + 1/q = 1 \), is semi-reflexive.

Proof: Consider the locally convex space \( \ell^p [t_\alpha(t^p)] \). Its dual is \( \ell^q \). On this dual, the strong topology \( \tau(q) \) is nothing but the usual normed topology \( \tau_\parallel \). Therefore, \( (\ell^q [t_\alpha(t^p)])^* = (\ell^p [t_\alpha(t^p)])^* = \ell^p \). It means \( \ell^p [t_\alpha(t^p)] \) is semi-reflexive.

2.4. Corollary: \( \ell^p [t_\alpha(t^p)] \), where \( 1 < p < \infty \) and \( 1/p + 1/q = 1 \), is polar semi-reflexive.

Proof: It follows from the fact that every semi-reflexive locally convex space is polar semi-reflexive ([1], §23.9(3)).

For \( p = 2 \), we obtain

2.5. Corollary: The locally convex space \( \ell^2 [t_\alpha(t^2)] \) is semi-reflexive.

Though the \( \text{(B)} \)-space \( \ell^1 \) (with the norm topology) is nonreflexive, but if we consider the space \( \ell^1 \) with the Mackey topology or the weak topology, then it holds some reflexivities as asserted in the following two theorems-

2.6. Theorem: The locally convex space \( \ell^1 [t_\alpha(c_0)] \) is inductively semi-reflexive.

Proof: Consider the locally convex space \( \ell^1 [t_\alpha(c_0)] \). Its dual is \( c_0 \). On this dual, the topology \( (t_\alpha(c_0))^* \) is nothing but the usual normed topology and therefore, \( (c_0 [t_\alpha(c_0)])^* = (c_0)^* = \ell^1 \).

Hence \( \ell^1 [t_\alpha(c_0)] \) is inductively semi-reflexive.

2.7. Corollary: The locally convex space \( \ell^1 [t_\alpha(c_0)] \) is semi-reflexive.

Proof: Inductively semi-reflexive locally convex space is always semi-reflexive, by ([2], (1.6)).

2.8. Theorem: The locally convex space \( \ell^1 [t_\alpha(c_0)] \) is semi-reflexive.

Proof: Consider the locally convex space \( \ell^1 [t_\alpha(c_0)] \). We have \( (\ell^1 [t_\alpha(c_0)])^* = c_0 \). On this dual, the strong topology \( \tau_\parallel \) is its norm topology. Therefore, \( (c_0 [t_\alpha(c_0)])^* = (c_0)^* = \ell^1 \) (see 1.5). Hence \( \ell^1 [t_\alpha(c_0)] \) is semi-reflexive. Using the fact that semi-reflexivity implies polar semi-reflexivity, we have

2.9. Corollary: The locally convex space \( \ell^1 [t_\alpha(c_0)] \) is polar semi-reflexive.

III. Conclusion

Each of the sequence space \( \ell^p \), \( 1 < p < \infty \), (and, in particular, \( \ell^1 \)) is both \( \text{B} \)-reflexive and inductively reflexive. On the other hand, on the dual \( c_0 \) of \( \ell^1 [t_\alpha(c_0)] \), the polar topology \( (t_\alpha(c_0))^* \) of uniform convergence on \( t_\alpha(c_0) \)-precompact subsets of \( \ell^1 \) is the usual normed topology. Now, in \( c_0 \), the set \( S = \{1, 1/2, \ldots, 1/n, \ldots\} \) is precompact for the normed topology and so \( (t_\alpha(c_0))^* \) precompact. But \( S \) is not finite dimensional. It implies that the topology \( (t_\alpha(c_0))^* \) of uniform convergence on \( t_\alpha(c_0) \)-precompact subsets of \( c_0 \) is strictly finer than \( t_\alpha(c_0) \). Therefore, \( \ell^1 [t_\alpha(c_0)] \) can’t be polar reflexive.

References