Orthogonal Generalized Derivations of Semiprime Semirings

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Abstract: Motivated by some results on Orthogonal Generalized Derivations of Semiprime Rings, in [2], the authors defined the notion of derivations and generalized derivations on semirings and investigated some results on the derivations in semirings. In this paper, we also introduce the notion of orthogonal generalized derivations of Semiprimesemirings and derived some interesting results.

**keywords:** Semirings, Derivations, Orthogonal derivations, Generalized orthogonal derivations, Centralizer

I. Introduction

This paper has been inspired by the work of Argac, Nakajima and Albas and also Mehsin Jabel Atteya\textsuperscript{[8]} and [7]. Throughout this paper S will represent a Semiring with the center Z(S). Bresar and vukman\textsuperscript{[1]} introduced the notation of Orthogonality for a pair d, g of derivations on a Semiprime Ring, and they contributed several necessary and sufficient conditions for d and g to be Orthogonal. Argac, Nakajima and Albas\textsuperscript{[5]} introduced the notation of Orthogonality generalized for a pair D, G of derivations on torsionfree Semiprime Ring, and extended the results of Orthogonal derivations to Orthogonal Generalized derivations. Majeed and Mehsin\textsuperscript{[6]} proved the following result in his paper, if R is a torsionfree Semiprime ring, (D, d) and (G, g) are generalized derivations of R such that R admits to satisfy \( \{d(x), g(x)\} = 0 \), for all \( x \in R \) and d acts as left centralizer (resp g acts as a left centralizer), then (D, d) and (G, g) are Orthogonal Generalized derivations of R. In this paper we study and investigate some interesting results concerning a non-zero generalized derivations with left cancellation property on Semiprimesemirings, when the non-zero additive mapping acts as a left centralizer of S.

II. Preliminaries

**Definition 2.1**

A **Semiring** \((S, +, \cdot)\) is a non-empty set S together with two binary operations, + and \cdot such that

1. \((S, +)\) and \((S, \cdot)\) are a Semigroup.
2. For all \(a, b, c \in S\), \(a \cdot (b + c) = a \cdot b + a \cdot c\) and \((b + c) \cdot a = b \cdot a + c \cdot a\)

**Definition 2.2:** A semiring S is said to be **2-torsionfree** if \(2x = 0 \Rightarrow x = 0, \forall x \in S\).

**Definition 2.3**

A semiring S is **Prime** if \(xS = 0 \Rightarrow x = 0, \forall x \in S\) and S is **Semi Prime** if \(xSx = 0 \Rightarrow x = 0, \forall x \in S\).

**Definition 2.4:** An additive mapping \(d : S \rightarrow S\) is called a **derivation** if \(d(xy) = d(x)y + xd(y), \forall x, y \in S\)

**Definition 2.5:** An additive mapping \(d : S \rightarrow S\) is called a **left centralizer** if \(d(xy) = d(x)y, \forall x, y \in S\).

**Definition 2.6:** Let d, g be two additive maps from S to S. They are said to be **Orthogonal** if \(d(x)Sg(y) = 0 = g(y)Sd(x), \forall x, y \in S\).

**Definition 2.7**

An additive mapping \(D : S \rightarrow S\) is called a **generalized derivation** if there exists a derivation \(d : S \rightarrow S\) such that \(D(xy) = D(x)y + xd(y), \forall x, y \in S\).

**Definition 2.8**

Two generalized derivations \((D, d)\) and \((G, g)\) of S are called **Orthogonal** if \(D(x)S G(y) = 0 = G(y)S D(x), \forall x, y \in S\)

We write \([x, y] = x y - y x\) and note that important identity \([x, z] = x[y, z] + [x, z]y\) and \([x, yz] = y[x, z] + [x, y]z\)
III. Orthogonal Generalized Derivations In Semirings

Lemma 3.1

Let S be a 2-torsion free Semiprime Semiring. Let D and G be two generalized derivations of S. If D and G are orthogonal to g and d respectively, then (i) dg = 0 and DG is a left centralizer of S. (ii) gd = 0 and GD is a left centralizer of S.

Proof:

(i) Since D and g are orthogonal, we get D(x)sg(y) = 0, ∀ x, y, s ∈ S.
Replacing x by xr we get D(xr)sg(y) = 0, ∀ x, y, r, s ∈ S.

⇒ [D(x)r + xd(r)] sg(y) = 0 ⇒ D(x)sg(y) + xd(r)sg(y) = 0

⇒ d(r)sg(y) = 0 (∵ D and g are orthogonal)
⇒ d(r)sg(y) = 0
⇒ g(y)sd(x) = 0.

∴ d and g are orthogonal.

Therefore dg = 0. Now we prove that DG is a left centralizer of S.

Since D is orthogonal to g and G is orthogonal to d we get, D(x)sg(y) = 0 and G(x)sd(y) = 0 so DG = 0 and Gd = 0, ∀ x, y, s ∈ S

Now DG(xy) = D[G(xy)] = D[G(x)y + xg(y)] = D[G(x)y] + D[xg(y)]
= DG(x)y + G(x)dg(y) + D(x)g(y) + xd(g(y)) = DG(x)y
(∵ by(1))

DG is a left centralizer of the Semiring. Similarly we shall prove (ii)

Lemma 3.2

Let S be a Semiprime Semiring, d a non-zero derivation of S, and U a non-zero left ideal of S. If for some positive integers t0, t1, ..., tn, and all x ∈ U, the identity
[(...[d(xn), x]t1, x]t2, ..., x]tn = 0 holds, then either d(U) = 0 or else d(U) and d(S)U are contained in a non-zero central ideal of S. In particular, if S is a prime semiring, then S is commutative

Theorem 3.3

Let S be a Semiprime Semiring with left Cancellation property, (D, d) and (G, g) be a non-zero generalizations of D and G respectively. If S admits to satisfy [d(x), g(x)] = 0, ∀ x ∈ U and a non-zero d acts as a left centralizer (resp a non-zero g acts as a left centralizer), then S contains a non-zero central ideal.

Proof: We have [d(x), g(x)] = 0, ∀ x ∈ S
Replacing x by xy, [d(xy), g(xy)] + [xd(y), g(xy)] = 0
⇒ d(xy), g(xy)] + [d(x), g(xy)]y + [x, g(xy)]d(y) = 0, ∀ x, y, s ∈ S
⇒ d(xy), g(xy)] + [d(x), g(xy)]y + [d(x), g(xy)]y + [d(x), g(xy)]y + [x, g(xy)]d(y) + [x, g(xy)]d(y) + [x, g(xy)]d(y) + [x, g(xy)]d(y) + [x, g(xy)]d(y) = 0, ∀ x, y, s ∈ S

Replacing y by x and according to the relation [d(x), g(x)] = 0
d(x)[x, g(x)] + d(x)[x, g(x)] + g(x)[d(x), x]x + [d(x), x][g(x) + xg(x)][d(x), x] + [x, g(x)]d(x) + [x, g(x)]d(x) + [x, g(x)]d(x) + [x, g(x)]d(x) = 0, ∀ x ∈ U
⇒ d(x)[x, g(x)] + d(x)[x, g(x)] + g(x)[d(x), x]x + [d(x), x][g(x) + xg(x)][d(x), x] + [x, g(x)]d(x) + [x, g(x)]d(x) + [x, g(x)]d(x) + [x, g(x)]d(x) = 0, ∀ x ∈ U
⇒ d(x)[x, g(x)] + d(x)[x, g(x)] + g(x)[d(x), x]x + [d(x), x][g(x) + xg(x)][d(x), x] + [x, g(x)]d(x) + [x, g(x)]d(x) + [x, g(x)]d(x) + [x, g(x)]d(x) = 0, ∀ x ∈ U
⇒ d(x)[x, g(x)] + d(x)[x, g(x)] + g(x)[d(x), x]x + [d(x), x][g(x) + xg(x)][d(x), x] + [x, g(x)]d(x) + [x, g(x)]d(x) + [x, g(x)]d(x) + [x, g(x)]d(x) = 0, ∀ x ∈ U
⇒ d(x)[x, g(x)] + d(x)[x, g(x)] + g(x)[d(x), x]x + [d(x), x][g(x) + xg(x)][d(x), x] + [x, g(x)]d(x) + [x, g(x)]d(x) + [x, g(x)]d(x) + [x, g(x)]d(x) = 0, ∀ x ∈ U
⇒ d(x)[x, g(x)] + d(x)[x, g(x)] + g(x)[d(x), x]x + [d(x), x][g(x) + xg(x)][d(x), x] + [x, g(x)]d(x) + [x, g(x)]d(x) + [x, g(x)]d(x) + [x, g(x)]d(x) = 0, ∀ x ∈ U

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\[ \text{Proof:} \] The theorem is nothing to prove if we replace \( d \) by \( D \) and \( g \) by \( G \) and use the generalization property in the above theorem.

Theorem 3.5

Let \( S \) be a SemiprimeSemiring with left Cancellation property, \( (D, d) \) and \( (G, g) \) be two non-zero generalized derivations of \( S \), and \( U \) a non-zero ideal of \( S \). If \( S \) admits to satisfy \( [D(x), G(x)] = 0, \forall x \in U \) a non-zero \( d \) acts as a left centralizer (resp a non-zero \( g \) acts as a left centralizer), then \( S \) contains a non-zero central ideal.

**Proof:** Given \( [D(x), G(x)] = [d(x), g(x)], \forall x \in U \) (1) Then \( [D(x), G(x)] = d(x)g(x) - g(x)d(x), \forall x \in U \) \( d \) acts as a left centralizer, \( [D(x), G(x)] = d(x)g(x) - g(x)d(x), \forall x \in U \) \( = d(x)g(x) + x^2d(x)g(x) - g(x)d(x), \forall x \in S \) \( [D(x), G(x)] = [d(x), g(x)] + x^2d(x)g(x), \forall x \in U \) (2) Right multiplying by \( y \) and since \( d \) acts as a left centralizer, \( xd(g(x)) \), \( y \) \( = 0, \forall x, y \in U \) \( xd(g(x)) + g(x)d(x) = 0, \forall x, y \in U \) By (2), \( x^2d(x)g(x) = 0, \forall x, y \in U \) Using the same method in Theorem 3:3, we complete our proof.

Theorem 3.6

Let \( S \) be a SemiprimeSemiring with left Cancellation property, \( (D, d) \) and \( (G, g) \) be two non-zero generalized derivations of \( S \), and \( U \) a non-zero ideal of \( S \). If \( S \) admits to satisfy \( [D(x), G(x)] = [d(x), g(x)], \forall x \in U \) and a non-zero \( d \) acts as a left centralizer (resp a non-zero \( G \) acts as a left centralizer), then \( S \) contains a non-zero central ideal.

**Proof:** Given \( [D(x), G(x)] = [d(x), g(x)], \forall x \in U \) Replacing \( x \) by \( xy \), \( D(xy)G(xy) - G(xy)D(xy) = [d(xy), g(xy)], \forall x, y \in U \) \( D(xy)G(xy) = D(xy)g(xy) - g(xy)D(xy) = [d(xy), g(xy)], \forall x, y \in U \) \( D(xy)G(xy) = D(xy)g(xy) - g(xy)D(xy) = [d(xy), g(xy)], \forall x, y \in U \) \( [D(xy), G(xy)] = [d(xy), g(xy)] + x^2D(xy)g(xy) = 0, \forall x, y \in U \) \( G(xy)[D(xy), y]D(xy)g(xy) + D(xy)g(xy) - g(xy)D(x)g(y) = [d(xy), g(xy)], \forall x, y \in U \) \( G(x)D(x)g(x) = D(x)g(x) - g(x)D(x)g(x) = [d(x), g(x)], \forall x, y \in U \) (1) Since \( D \) acts as a left centralizer
By using left cancellation property of $xg(x)x$ then $xg(x)x = 0$.

Substituting (1) in (3),

$$D(x)xG(x) + D(x)x = 0$$

Since $g$ acts as a left centralizer, (2) becomes

$$D(x)g(x)x + d(x)g(x)x + xd(x)g(y) - xg(y)D(x) = [d(x), g(x)]$$

Using (1), $xd(y)xg(y) = 0, \forall x, y \in U$

By the same argument used in theorem 3.3, we complete our proof.

**Theorem 3.7**

Let $S$ be a Semiprime Semiring with Cancellation property, $(D, d)$ and $(G, g)$ be two generalized derivations of $S$, and $U$ a non-zero ideal of $S$. If $S$ admits to satisfy $[D(x), G(x)] = [d(x), g(x)], \forall x \in U$ and $D$ and a non-zero $g$ act as a left centralizers (resp $g$ and a non-zero $d$ acts as a left centralizers), then $S$ contains a non-zero central ideal.

**Proof:** Given $[D(x), G(x)] = [d(x), g(x)], \forall x \in U$

Replacing $x$ by $xy$ and Since $D$ and $g$ acts as a left centralizers, we obtain

$$D(x)g(x)x + D(x)x + G(x)x = [d(x), g(x)]x$$

Using (1), $xd(y)xg(y) = 0, \forall x, y \in U$

Then $D(x)xG(x)x + D(x)x + G(x)x = 0, \forall x, y \in U$

The same argument used in theorem 3.3, we complete our proof.

**References**


