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Stability Analysis of Thin Liquid Film by Long-Wave Method

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Abstract:*stability and dynamics of a free double-sided symmetric thin liquid film are investigated by using the long-wave method. The flow in thin liquid film is considered in two dimensions for Newtonian liquid with constant density and dynamic viscosity. The Navier-stokes equations is used with appropriate boundary conditions of zero shear stress and also of normal stress on the bounding free surfaces with non-dimensional variables to obtain an equation that governs such flow.*
Keywords:*Thin Liquid Films, Navier-Stokes equations, continuity equation.*

I. Introduction

This paper is to study the linear stability analysis of thin liquid symmetrical double-sided film. The film may be modeled by a two dimensional flow of a Newtonian liquid of constant density and constant viscosity. The equations of continuity and motion with appropriate boundary conditions are solved taking into account the variation of the local surface tension. A long-wave approximation applied to those governing equations lead to an equation of evolution of the film as function of time and space. The stability and dynamics of thin liquid films, in general, are importance in many applications. [1] Considered the stability analysis of an inclined free thin liquid film and this equation affected by hydrostatic pressure and surface tension.

The stability of thin fluid films is essential in applications, such as in coatings (paints), photographic films, in microelectronic devices and insulating layers [2]. The stability analysis are considered by [3] which given the critical number, the wave length, and the maximum growth rate of the most unstable disturbance. In this paper, we consider the flow in a double-sided horizontal thin liquid film. We use the Navier- stokes equation and, we obtain the equation that governs such flow with negligible inertia term. And the flow is predominantly in X- derction. The stability analysis is investigated by using the long-wave theory.

II. Mathematical Formulation

We assume here two dimensional incompressible flow governed by the Navier-Stokes equations of motion in X and Ydirections for unsteady flow for thin liquid film as shown in Fig. (1). The Nanier-stokes equations are given by

$$\rho\left[\frac{\partial U}{\partial T}+U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}\right]=-\frac{\partial P}{\partial X}+\mu\left[\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right] \qquad (1)$$

and

$$\rho\left[\frac{\partial V}{\partial T}+U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}\right]=-\frac{\partial P}{\partial Y}+\mu\left[\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right] \qquad (2)$$

whereT is the time,P is the pressure, μ is the dynamic viscosity of the fluid and ρ is the density of the liquid, and a Cartesian coordinate system (X, Y) with corresponding velocity components (U, V) is used.
The continuity equation is given by

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y} \qquad (3)$$

In lubrication theory, the inertia terms can be neglected and the Navier-Stokes equations (1) and (2) becomes

$$\frac{\partial P}{\partial X}=\mu\left[\frac{\partial^2 U}{\partial X^2}+\frac{\partial^2 U}{\partial Y^2}\right] \qquad (4)$$

and

$$\frac{\partial P}{\partial Y}=\mu\left[\frac{\partial^2 V}{\partial X^2}+\frac{\partial^2 V}{\partial Y^2}\right] \qquad (5)$$

Now from the conservation of mass and since the free surface is a stream line, the material or the substantial derivative$\frac{DF}{DT}=0$ must be vanished on $Y=H(X,T)$and thus, we have

$$\frac{DF}{DT}=\frac{\partial F}{\partial T}+U\frac{\partial F}{\partial X}+V\frac{\partial F}{\partial Y}=0 \qquad (6)$$

where$F(X,Y,T)=Y-H(X,T)$ (7)

From equations (6) and (7), we have

$$\frac{\partial H}{\partial T}+U\frac{\partial H}{\partial X}-V=0 \qquad (8)$$

For this squeezing mode, the perturbations of the two surfaces are symmetric [4] and thus:
$+H(X,T)=-H(X,T)$.

Here after, we write the dimensional thickness as $+\mathrm{H}(\mathrm{X},\mathrm{T}) \equiv \mathrm{H}(\mathrm{X},\mathrm{T})$ and will consider only the upper part of the symmetric film $0 \leq \mathrm{Y} \leq \mathrm{H}(\mathrm{X},\mathrm{T})$.
The conditions for the squeezing mode ([4] and [5]) of the free film are then:

$$\mathrm{V} = 0 \text{ at } \mathrm{Y} = 0 \,. \tag{9}$$

III. Dimensional Analysis

To express the Navier-Stokes equations, the equation of continuity and the associated boundary conditions into non-dimensional form, We now introduce non-dimensional variable as follows [6],

$$\mathrm{x} = \frac{\mathrm{k}\epsilon\mathrm{X}}{\overline{\mathrm{H}}}, \qquad \mathrm{y} = \frac{\mathrm{Y}}{\overline{\mathrm{H}}}, \qquad \mathrm{h} = \frac{\mathrm{H}}{\overline{\mathrm{H}}}, \qquad \mathrm{u} = \frac{\mathrm{U}}{\overline{\mathrm{U}}}$$

$$\mathrm{v} = \frac{\mathrm{V}}{\epsilon\overline{\mathrm{U}}}, \qquad \mathrm{p} = \frac{\overline{\mathrm{H}}\mathrm{P}}{\epsilon^2\sigma}, \mathrm{t} = \frac{\epsilon\overline{\mathrm{U}}\mathrm{T}}{\overline{\mathrm{H}}} \tag{10}$$

Where $\overline{\mathrm{H}}$ and $\overline{\mathrm{U}}$ are the characteristics, and a Capillary number is

$$\epsilon = \frac{\mu\overline{\mathrm{U}}}{\sigma} \tag{11}$$

Also a Reynolds number is

$$\mathrm{k}\,\epsilon = \frac{\rho\overline{\mathrm{U}}\overline{\mathrm{H}}}{\mu} \tag{12}$$

where ϵ and k are non-dimensional constants.
Substituting equation (11) into equation (12) , we obtain

$$\mathrm{k} = \frac{\rho\sigma\overline{\mathrm{H}}}{\mu^2} \tag{13}$$

By using the dimensionless variables given by equations (10), into equations (3)-(5), we havethe continuity equation gives the form

$$\mathrm{k}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \frac{\partial \mathrm{v}}{\partial \mathrm{y}} = 0 \tag{14}$$

and the Navier-Stoke equation in x and y directions respectively give

$$\frac{\partial^2 \mathrm{u}}{\partial \mathrm{y}^2} = \epsilon^2\left[\mathrm{k}\frac{\partial \mathrm{p}}{\partial \mathrm{x}} - \mathrm{k}^2\frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2}\right] \tag{15}$$

$$\frac{\partial^2 \mathrm{v}}{\partial \mathrm{y}^2} = \left[\frac{\partial \mathrm{p}}{\partial \mathrm{y}} - \mathrm{k}^2\,\epsilon^2\,\frac{\partial^2 \mathrm{v}}{\partial \mathrm{x}^2}\right] \tag{16}$$

and equation (8) in non-dimensional gives the form

$$\mathrm{v} = \frac{\partial \mathrm{h}}{\partial \mathrm{t}} + \mathrm{k}\mathrm{u}\frac{\partial \mathrm{h}}{\partial \mathrm{x}} \tag{17}$$

The shear-stress condition and the normal-stress condition by [6] in non-dimensional form respectively given as

$$\frac{\partial \mathrm{u}}{\partial \mathrm{y}} + \epsilon^2\,\mathrm{k}\frac{\partial \mathrm{v}}{\partial \mathrm{x}} - 4\,\epsilon^2\,\mathrm{k}^2\frac{\partial \mathrm{h}}{\partial \mathrm{x}}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} = 0 \tag{18}$$

$$\mathrm{p} = 2\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + 2\frac{\partial \mathrm{h}}{\partial \mathrm{x}}\left[\frac{\partial \mathrm{u}}{\partial \mathrm{y}} + \epsilon^2\,\frac{\partial \mathrm{v}}{\partial \mathrm{x}}\right] = -\frac{\partial^2 \mathrm{h}}{\partial \mathrm{x}^2} \tag{19}$$

and the equation (9) in non-dimensional gives the form

$$\mathrm{v} = 0 \text{ at } \mathrm{y} = 0 \tag{20}$$

IV. Long-Wave Expansion Theory

Long-wave expansion theory is an effective means of revealing the nonlinear evolution equation of film thickness. [7], indicated that long-wave perturbation is appropriate for film flow with a small Reynolds number. The wave numberk_1 can be treated as a perturbed parameter.
We now present the deviation procedure demonstrated by [8]. The new dependent and independent variables in the system are defined as:

$$\mathrm{x} = {\mathrm{k}_1}^{-1}\xi, \qquad \mathrm{y} = \zeta, \qquad \mathrm{t} = {\mathrm{k}_1}^{-2}\tau$$

$$\mathrm{h} = \mathrm{H}_2, \qquad \mathrm{u} = \mathrm{k}_1\mathrm{U}^*, \qquad \mathrm{v} = {\mathrm{k}_1}^2\mathrm{V}^*$$

$$\mathrm{p} = {\mathrm{k}_1}^2\mathrm{P}^*, \tag{21}$$

Here, the uppercase variables and derivatives with respect to Greek variables are unit order as $\mathrm{k}_1 \to 0$.
We now seek solutions of the form

$$(\mathrm{U}^*,\mathrm{V}^*,\mathrm{P}^*) = (\mathrm{U}_0,\mathrm{V}_0,\mathrm{P}_0) + {\mathrm{k}_1}^2(\mathrm{U}_1,\mathrm{V}_1,\mathrm{P}_1) + {\mathrm{k}_1}^4(\mathrm{U}_2,\mathrm{V}_2,\mathrm{P}_2) + \cdots \tag{22}$$

Where,$\mathrm{H}_2 = O(1)$. We equate to zero coefficients of like powers of ${\mathrm{k}_1}^2$ and obtain a sequence of problems.
For k=1, we get $\overline{\mathrm{H}} = \frac{\mu^2}{\rho\sigma}$ and this ensures the balance between viscous forces and surface tension forces and then $\overline{\mathrm{H}}$ can be determined from the physical parameters for each liquid. Then, for others values of k, we can proceed in a similar way.
Now, by substituting equation (21) into equations (14)-(16), we obtain:

$$\frac{\partial(k_1 U^*)}{\partial(k_1^{-1}\xi)} + \frac{\partial(k_1^2 V^*)}{\partial\zeta} = 0 \quad , \tag{23}$$

$$\epsilon^2 \frac{\partial(k_1^2 P^*)}{\partial(k_1^{-1}\xi)} = \epsilon^2 \frac{\partial}{\partial(k_1^{-1}\xi)}\frac{\partial(k_1 U^*)}{\partial(k_1^{-1}\xi)} + \frac{\partial}{\partial\zeta}\frac{\partial(k_1 U^*)}{\partial\zeta} \tag{24}$$

and

$$\frac{\partial(k_1^2 P^*)}{\partial\zeta} = \epsilon^2 \frac{\partial}{\partial(k_1^{-1}\xi)}\frac{\partial(k_1^2 V^*)}{\partial(k_1^{-1}\xi)} + \frac{\partial}{\partial\zeta}\frac{\partial(k_1^2 V^*)}{\partial\zeta} \tag{25}$$

Also substituting equation (21) into equations (17)-(20), we obtain:

$$k_1^2 V^* = \frac{\partial H_2}{\partial(k_1^{-2}\tau)} + k_1 U^* \frac{\partial H_2}{\partial(k_1^{-1}\xi)} \tag{26}$$

$$\frac{\partial(k_1 U^*)}{\partial\zeta} + \epsilon^2 \frac{\partial(k_1^2 V^*)}{\partial(k_1^{-1}\xi)} - 4\,\epsilon^2 \frac{\partial H_2}{\partial(k_1^{-1}\xi)}\frac{\partial(k_1 U^*)}{\partial(k_1^{-1}\xi)} = 0 \tag{27}$$

$$k_1^2 P^* + 2\frac{\partial(k_1 U^*)}{\partial(k_1^{-1}\xi)} + 2\frac{\partial H_2}{\partial(k_1^{-1}\xi)}\left[\frac{\partial(k_1 U^*)}{\partial\zeta} + \epsilon^2 \frac{\partial(k_1^2 V^*)}{\partial(k_1^{-1}\xi)}\right]$$

$$= \frac{\partial}{\partial(k_1^{-1}\xi)}\frac{\partial H_2}{\partial(k_1^{-1}\xi)} \tag{28}$$

$$k_1^2 V^* = 0 \text{at} y = 0 \tag{29}$$

After simplifying the above equations, we can write thecontinuity equation and the Navier-Stokes equations in the following form:

$$k_1^2 \frac{\partial U^*}{\partial\xi} + k_1^2 \frac{\partial V^*}{\partial\zeta} = 0 \tag{30}$$

$$\epsilon^2 k_1^3 \frac{\partial P^*}{\partial\xi} = \epsilon^2 k_1^3 \frac{\partial^2 U^*}{\partial\xi^2} + k_1 \frac{\partial^2 U^*}{\partial\zeta^2} \tag{31}$$

$$k_1^2 \frac{\partial P^*}{\partial\zeta} = \epsilon^2 k_1^4 \frac{\partial^2 V^*}{\partial\xi^2} + k_1^2 \frac{\partial^2 V^*}{\partial\zeta^2} \tag{32}$$

And the boundary conditions (26)-(29), becomes

$$k_1^2 V^* = k_1^2 \frac{\partial H_2}{\partial\tau} + k_1^2 U^* \frac{\partial H_2}{\partial\xi} \text{at} y = \pm h \tag{33}$$

$$k_1 \frac{\partial U^*}{\partial\zeta} + \epsilon^2 k_1^3 \frac{\partial V^*}{\partial\xi} - 4\,\epsilon^2 k_1^3 \frac{\partial H_2}{\partial\xi}\frac{\partial U^*}{\partial\xi} = 0 \text{at} y = \pm h \tag{34}$$

$$k_1^2 P^* + 2k_1^2 \frac{\partial U^*}{\partial\xi} + 2k_1 \frac{\partial H_2}{\partial\xi}\left[k_1 \frac{\partial U^*}{\partial\zeta} + \epsilon^2 k_1^3 \frac{\partial V^*}{\partial\xi}\right] = -k_1^2 \frac{\partial^2 H_2}{\partial\xi^2} \text{at} y = \pm h \tag{35}$$

and

$$k_1^2 V^* = 0 \text{at} y = 0 \tag{36}$$

At leading order, we obtain:

$$\frac{\partial U_0}{\partial\xi} + \frac{\partial V_0}{\partial\zeta} = 0 \tag{37}$$

$$\frac{\partial^2 U_0}{\partial\zeta^2} = 0 \tag{38}$$

$$\frac{\partial^2 V_0}{\partial\zeta^2} - \frac{\partial P_0}{\partial\zeta} = 0 \tag{39}$$

$$V_0 = \frac{\partial H_2}{\partial\tau} + U^* \frac{\partial H_2}{\partial\xi} \text{at} \zeta = H_2 \tag{40}$$

$$\frac{\partial U_0}{\partial\zeta} = 0 \text{at} \zeta = H_2 \tag{41}$$

$$P_0 + 2\frac{\partial U_0}{\partial\xi} + 2\frac{\partial H_2}{\partial\xi}\frac{\partial U_0}{\partial\zeta} + \frac{\partial^2 H_2}{\partial\xi^2} = 0 \text{at} \zeta = H_2 \tag{42}$$

$$V_0 = 0 \text{at} \zeta = 0 \tag{43}$$

The solution of equation (38) is given by:

$$\frac{\partial U_0}{\partial\zeta} = E(\xi, \tau) \tag{44}$$

from the boundary condition (41), we get:

$$E(\xi, \tau) = 0$$

Then by integrating equation (44) with respecttoζ, we obtain:

$$U_0 = C(\xi, \tau) \tag{45}$$

where C is still unknown function of ξ and τ.Integrated equation (37) with respect to ζ, and using equation (44), we have:

$$V_0 = -\frac{\partial C}{\partial\xi}\zeta + l(\xi, \tau) \tag{46}$$

By using the condition (43), the above equation becomes

$$V_0 = -\frac{\partial C}{\partial\xi}\zeta \tag{47}$$

Equation (39) can be integrated once with respect toζ, we get

$P_0 = \frac{\partial V_0}{\partial \zeta} + D(\xi, \tau)$ (48)

where D is another unknown function of ξ and τ . Since P_0 is independent of ζ, P_0 is given by normal stress condition (42). Then by using equation (42) and from (48), we can determine the function D as :

$$-2\frac{\partial U_0}{\partial \xi} - 2\frac{\partial H_2}{\partial \xi}\frac{\partial U_0}{\partial \zeta} - \frac{\partial^2 H_2}{\partial \xi^2} = \frac{\partial V_0}{\partial \zeta} + D(\xi, \tau)$$

or

$D(\xi, \tau) = -\frac{\partial C}{\partial \xi} - \frac{\partial^2 H_2}{\partial \xi^2}$ (49)

Finally, bysubstituting U_0 and $V_0$from equations (45) and (47),into the equation (40) to obtain:

$-\frac{\partial C}{\partial \xi}\zeta = \frac{\partial H_2}{\partial \tau} + C\frac{\partial H_2}{\partial \xi}\text{at}\zeta = H_2$ (50)

or

$\frac{\partial H_2}{\partial \tau} + \frac{\partial (H_2 C)}{\partial \xi} = 0$ (51)

Since both C and H_2 are unknown, a second relation between the unknown functions C and H_2 must be obtained. To this end, we analyze the $O({k_1}^2)$ problem forU_1, from equation (31), we have:

$\frac{\partial^2 U_1}{\partial \zeta^2} = \epsilon^2 \frac{\partial P_0}{\partial \xi} - \epsilon^2 \frac{\partial^2 U_0}{\partial \xi^2}$ (52)

and

$\frac{\partial U_1}{\partial \zeta} = 0\text{at}\zeta = 0$ (53)

From the sheer-stress condition (34), we obtain:

$\frac{\partial U_1}{\partial \zeta} + \epsilon^2 \frac{\partial V_0}{\partial \xi} - 4\,\epsilon^2 \frac{\partial U_0}{\partial \xi}\frac{\partial H_2}{\partial \xi} = 0\text{at}\zeta = H_2$ (54)

By using the expressions forU_0, V_0 and P_0 given by equations (45), (47) and (48),the equations (52)-(54), becomes:

$\frac{\partial^2 U_1}{\partial \zeta^2} = \epsilon^2 \left[\frac{\partial D}{\partial \xi} - \frac{\partial^2 C}{\partial \xi^2} - \frac{\partial^2 C}{\partial \xi^2}\right]$ (55)

$\frac{\partial U_1}{\partial \zeta} = 0\text{at}\zeta = 0$ (56)

$\frac{\partial U_1}{\partial \zeta} - \epsilon^2 \left[\frac{\partial^2 C}{\partial \xi^2}H_2 + 4\frac{\partial C}{\partial \xi}\frac{\partial H_2}{\partial \xi}\right] = 0\text{at}\zeta = H_2$ (57)

We integrate equation (55) once and use equation (56) to obtain:

$\frac{\partial U_1}{\partial \zeta} = \epsilon^2 \left[\frac{\partial D}{\partial \xi} - \frac{\partial^2 C}{\partial \xi^2} - \frac{\partial^2 C}{\partial \xi^2}\right]\zeta$ (58)

Now, from equations (57) and (58), we have:

$\frac{\partial^2 C}{\partial \xi^2}H_2 + 4\frac{\partial C}{\partial \xi}\frac{\partial H_2}{\partial \xi} = \left[\frac{\partial D}{\partial \xi} - 2\frac{\partial^2 C}{\partial \xi^2}\right]H_2$ (59)

Using equation (49), we eliminate D from equation (59) and obtain the following equation for C and H_2 :

$$\frac{\partial^2 C}{\partial \xi^2}H_2 + 4\frac{\partial C}{\partial \xi}\frac{\partial H_2}{\partial \xi} = \left[-\frac{\partial^2 C}{\partial \xi^2} - \frac{\partial^3 H_2}{\partial \xi^3} - 2\frac{\partial^2 C}{\partial \xi^2}\right]H_2$$

$-\frac{\partial^3 H_2}{\partial \xi^3}H_2 = 4\left[\frac{\partial\left(H_2\frac{\partial C}{\partial \xi}\right)}{\partial \xi}\right]$ (60)

We have found two coupled nonlinear evolution for the longitudinal component of velocity and the thickness of the film. We rewrite these equations in terms of the original variable by writing:

$\xi = k_1 x$, $\tau = {k_1}^2 t$,
$H_2 = h$, $C = {k_1}^{-1}u$ (61)

By substituting the equation (61) into equations (51) and (60), we obtain:

$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0$ (62)

and

$-\frac{\partial^3 h}{\partial x^3}h = 4\left[\frac{\partial\left(h\frac{\partial u}{\partial x}\right)}{\partial x}\right]$ (63)

V. Linear Stability Analysis

It is instructive to analyze the stability of the steady state solution [9]

$$(u, h) = \left(0, \frac{1}{2}\right)$$

By introducing the deviation:

$$h' = h + \frac{1}{2}$$

We obtain from equations (62) and (63) the following linearized problem:

$$\frac{\partial h'}{\partial t} + \frac{1}{2}\frac{\partial u}{\partial x} = 0 \qquad (64)$$

and

$$\frac{\partial^3 h'}{\partial x^3} + 4\frac{\partial^2 u}{\partial x^2} = 0 \qquad (65)$$

where h' represents the initial disturbance of the film thickness.Furthermore, the perturbed state of velocity and film thickness could be represented in the form of normal mode disturbance [10] and [11]:

$$(h', u) = (H_0, U_0)\exp(vt + ik_1x) \qquad (66)$$

where H_0 and U_0 denotes the amplitude of the initial disturbance. The characteristic equation is derived by substituting equation (66) into equations (64) and (65), we obtain:

$$H_0 v\exp(vt + ik_1x) + \frac{1}{2}iU_0k_1\exp(vt + ik_1x) = 0 \qquad (67)$$

and

$$-iH_0{k_1}^3\exp(vt + ik_1x) - 4U_0{k_1}^2\exp(vt + ik_1x) = 0 \qquad (68)$$

Since $\exp(vt + ik_1x) \neq 0$, then equations (67) and (68) becomes:

$$H_0v + \frac{1}{2}iU_0k_1 = 0 \qquad (69)$$

and

$$iH_0{k_1}^3 + 4U_0{k_1}^2 = 0 \qquad (70)$$

Using equation (69) into equation (70), we get the dispersion relation as:

$$v + \frac{{k_1}^2}{6} = 0 \qquad (71)$$

where v and k_1 denote the complex growth rate and real wave number, respectively. Solving the above relation with respect to v , we have:

$$v = -\frac{{k_1}^2}{6} \qquad (72)$$

Therefore, the film becomes unstable [3], viz., $v > 0$ only when $k_1 < k_c$ where k_c is a critical wave number. From equation (72), and since v is always negative for any value of k_1, and $k_1 < k_c$, the film becomes stable. For neutrally stable wave (i.e., $v = 0$),the critical wave number k_c is given by:

$$k_c = 0 \qquad (73)$$

The maximum growth rate v_m of linear waves occurs for the fastest growing (dominant) wave number, k_m which is obtained by setting $\frac{\partial v}{\partial k_1} = 0$ from equation (3.3.9), thus:

$$ {k_m}^2 = 0 \qquad (74)$$

VI. Figures

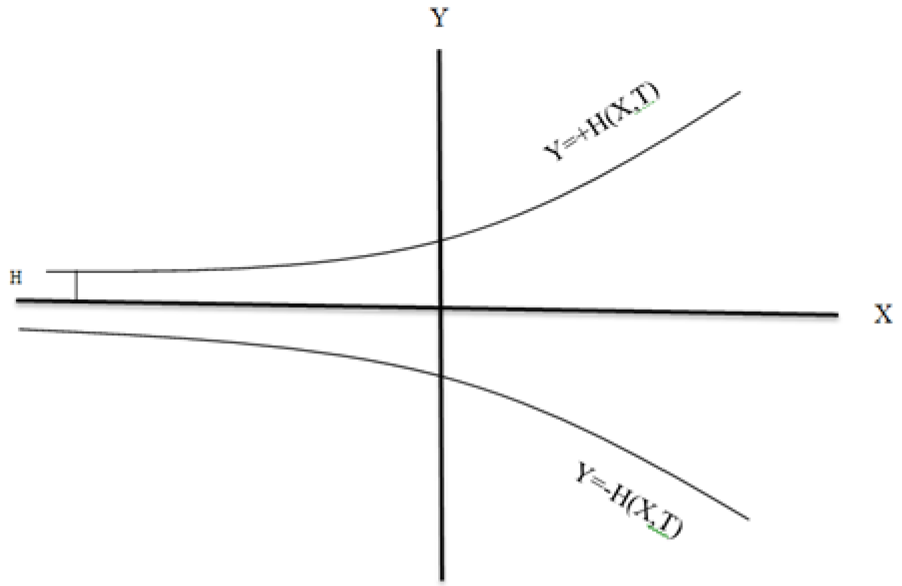

Fig. (1): cross- section of a horizontal symmetric thin liquid film.

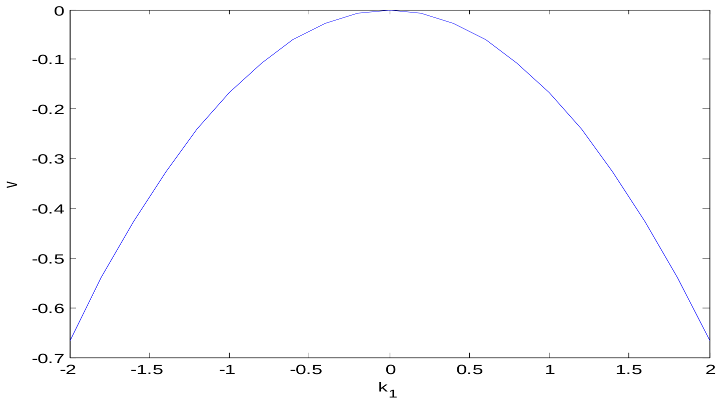

Fig. (2): the growth rate v vs. wave number k_1 plotted with unsteady term from equation (72).

VII. Conclusion

The stability of thin liquid symmetric double-sided film modeled in a two dimensional flow is constructed and it is seen from equation (72), that the film becomes stable, since v is always negative for any value of $k_1$and $k_1 < k_c$ .Furthermore when $v = 0$, the film becomes neutrally stable wave and also the critical wave number k_c and the maximum growth rate v_m of linear waves.

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