Fuzzy and Rough Approximations Operations on Graphs

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Abstract: In this paper, we introduce a covering model on connected graph for modified lower and upper approximations operation on graph vertices with respect to connected sub graph. We obtain new classes induced by covering every sub graphs by lower and upper approximation operations using fuzzification set of vertices of main graph. Also, properties of these classes studied and illustrate that with examples.

Keywords: Graph theory, Rough set theory, Topology, Fuzzy set theory and Data mining.

I. Introduction

Many extensions of rough set theory have been discussed for approximations methods in information systems and other applications. Ordinary rough set is a new approach proposed by Pawalak [11,18,19,20,21] to deal with incomplete or imperfect information.

Rough set analysis is based on relation that describes in distinguish ability of objects. The concepts are represented by their lower and upper approximations. In applications, rough sets focus on approximate reasoning. Rough sets are applied to data mining, machine learning, finance, industry, multimedia, medicine, control theory, pattern recognition, and most recently bioinformatics[5,11]. Rough set is non-statistical tool for analysis of imperfect Data, data dependencies, fuzzy set and decision rules [3,4,5,8,12,24].

Graph theory plays an important core for transformation any engineering, chemical, physical, social sciences, game theories, networks, preferences, and mathematication of discrete spaces. We may find new method for classification vertices of graph G=(V,E), where V is the set of vertices and E the set of edges, elements of the V depended on the form of link each vertex in V with others. this form can determined by neighborhoods which classified V to sets have elements are the same or belong to the same elementary set if they have the same link value.

In [17,18] neighborhoods of graph vertices used to studied some inequalities, which joint the number of each incident edge and number of vertices in open sets and closed sets in topological structures formed by this neighborhoods, then applied this on blood circulation in the human body and nervous system in some medical application of neighborhoods of graph vertices. This paper may be help us to knows the maximum and minimum numbers of vertices used to determined any path in connected graph with respect to part of this graph. We introduce some important new properties of graph in sense of rough set concepts, we also tried to used classification by neighborhoods of each vertices to obtained upper and lower sub graph of any part of graph by new mathematical methods.

This paper is organized as follows: In Sec.2 we introduced the main concepts of rough set theory. By neighborhoods on connected graph. In Sec.3 we introduced new definitions of lower and upper approximation operators by membership function and precision rough set model.

II. Rough Set Concepts On Connected Graph

We try investigate rough set concepts on vertices of graph as neighbors[3,8,9,15,17]. Upper and lower approximate of connected sub graph vertices with respect to the main graph determined by neighborhoods, which cover every sub graph of main graph, lower approximation concept make all vertices and edges lies completely in sub graph, upper approximation concept make all vertices and edges possibly lies in sub graph. some new rough concepts on this classes discussed.

Definition 2.1
Let G≡(V,E) be connected graph, the neighborhood of any vertex is defined as

\[ N(v_i) = \{v_i\} \cup \{v_j : v_j \in V(G), v_i v_j \in E\} \]

. The covering graph is defined as

\[ C(G) = \{G(N_{v_i}), v_i \in V(G), v_i v_j \in E(G)\} \]

. The minimal neighborhood of any vertex is defined as

\[ MN(v_i) = \cap \{G(N_{v_i}) : G(N_{v_i}) \in C(G), v_i \in G(N_{v_i})\} \]
Definition 2.2
Let \( G_1 \equiv (V_1, E_1) \) be a connected subgraph of a connected graph \( G \equiv (V, E) \), covering lower and covering upper approximation operation are defined on the graph as follows

\[ L_{G_1}, U_{G_1} : P(V) \to P(V) \]

\[ L_{G_1}(X) = \{ y : y \in MN(v_i) \subseteq X, \overline{v_i} \in E_1, X \subseteq V \text{ and } v_i \in V \} \]

\[ U_{G_1}(X) = \{ y : y \in MN(v_i) \cap X \neq \emptyset, \overline{v_i} \in E_1, X \subseteq V \text{ and } v_i \in V \} \]

Definition 2.3
Let \( G_1 \equiv (V_1, E_1) \) be a connected subgraph of a connected graph \( G=(V,E) \). Rough degree of \( G \) with respect to \( G \equiv (V, E) \), denoted by \( \rho_{G_1}(X) \), is defined as

\[ \rho_{G_1}(X) = 1 - \frac{L_{G_1}(X)}{U_{G_1}(X)} \]

Definition 2.4
Let \( G \equiv (V, E) \) be a simple connected graph and \( G_1 \equiv (V_1, E_1), G_2 \equiv (V_2, E_2) \) are subgraphs of \( G \).

The positive region, negative region and boundary region of \( G_2 \) with respect to \( G_1 \) define as

\[ pos_{G_1}(G_2) = L_{G_1}(V_2) \]

\[ neg_{G_1}(G_2) = V - U_{G_1}(V_2) \]

\[ bd_{G_1}(G_2) = U_{G_1}(V_2) - L_{G_1}(V_2) \]

Example 2.1: Consider the Connected graph \( G=(V,E) \)

The neighborhoods of vertices are

\[ N(a) = \{a, b, d\}, N(b) = \{b, c\}, N(c) = \{a, c, d\}, N(d) = \{b, d\} \]

The graphs of the neighborhoods are connected sub graphs of \( G=(V,E) \).
Then the covering of graph is $C(G) = \{G(N_a), G(N_b), G(N_c), G(N_d)\}$ and

$MN(a) = \{a\}, MN(b) = \{b\}, MN(c) = \{c\}, MN(d) = \{d\}$. Consider the sub graph $G_1(V_1, E_1)$

![Graph G1](image)

Fig (2.3)

The set of sub graph is $V_1 = \{a, b, c\}$. The covering lower approximation of each set of vertices in $G_1(V_1, E_1)$ with respect to sets of neighborhoods of all vertices of $G=(V,E)$ are formed as:

$L_{G_1}(\{a\}) = \{a\}, L_{G_1}(\{b\}) = \{b\}, L_{G_1}(\{c\}) = \{c\},$

$L_{G_1}(\{a, c\}) = \{a, c\}, L_{G_1}(\{a, b\}) = \{a, b\},$

$L_{G_1}(\{b, c\}) = \{b, c\}, L_{G_1}(V_1) = \{a, b, c\}.$

The covering upper approximation of each set of vertices in $G_1(V_1, E_1)$ with respect to sets of neighborhoods of all vertices of $G=(V,E)$ are formed as:

$U_{G_1}(\{a\}) = \{a\}, U_{G_1}(\{b\}) = \{b\}, U_{G_1}(\{c\}) = \{c\},$

$U_{G_1}(\{a, c\}) = \{a, c\}, U_{G_1}(\{a, b\}) = \{a, b\},$

$U_{G_1}(\{b, c\}) = \{b, c\}, U_{G_1}(V_1) = \{a, b, c\}.$

![Graph G2](image)

Fig (2.4)

$pos_{G_1}(G_2) = \{b, c, d\}, neg_{G_1}(G_2) = \{d\}$

We can obtain the upper and lower approximation of any path vertices in $G_1 \equiv (V_1, E_1)$ with respect to $G \equiv (V, E)$. Let $P_1=ae$ be a path from $a$ to $c$ so, the set of vertices of the path is $V(P_1) = \{a, b, c\}$ has lower and upper approximations as: $L_{G_1}(V(P_1)) = \{a, b, c\}, U_{G_1}(V(P_1)) = \{a, b, c\}$.

From above we notice that the previous of determine can apply the above definition on undirected connected graph by use the direction of every edge in two direction.

**Example 2.2:** Consider the undirected connected graph

![Graph G](image)

Fig (2.5)
$N(a) = V, N(b) = \{a, b, c\}, N(c) = V, N(d) = \{a, c, d\}$

**Fig (2.6)**

$C(G) = \{G(N_a), G(N_b), G(N_c), G(N_d)\}$, $MN(a) = \{a, c\}, MN(b) = \{a, b, c\}, MN(c) = \{a, c\}$ and $MN(d) = \{a, c, d\}$

$a$

$b$

d

e

c

$G_1 \equiv (V_1, E_1)$

$G_2 \equiv (V_2, E_2)$

**Fig (2.7)**

III. Rough Set Based On Rough Membership Function

In this section we introduce the concept of rough set theory by fuzzifications crisp set of sub graphs vertices using its neighborhoods $N_G(v_i)$ vertices, we obtained rough membership function indicated the dependence of graph $G=(V,E)$ vertices with sub graphs neighborhoods. The important question be , what is the different between graph operations (deletions, adding, unions, …) and the same operations on the new classes and parameters. To answer this question we must introduced some new results.

**Definition 3.1**

Let $G=(V,E)$ and $G_1=(V_1,E_1)$ be a subgraph. Then the membership function of any vertex with respect to the sub graph $G_1$ is defined as

$$
\mu^G_{G_1}(v_i) = \frac{\bigcap_{v_j} N_G(v_j) \cap V(G_1)}{|\bigcap_{v_j} N_G(v_j)|}, v_i \in N_G(v_i) \subseteq V(G)
$$

We can redefine the definition of lower and upper covering by using rough membership function.

**Proposition 3.1**

Let $G=(V,E)$ be a simple connected graph and $G_1=(V_1,E_1), G_2=(V_2,E_2)$ are subgraphs of $G$ and $G_2 \subseteq G_1$ then $\mu^G_{G_1}(v_i) \leq \mu^G_{G_2}(v_i)$ for all $v_i \in V(G)$

**Proof**

From the definition of membership and $|\bigcap N_G(v_i) \cap V(G_2)| \leq |\bigcap N_G(v_i) \cap V(G_1)|$
We obtained $\mu_{G}^{E}(v_{i}) \leq \mu_{G'}^{E}(v_{i})$

**Proposition 3.2**

Let $G \equiv (V,E)$ be a simple connected graph, $G_{1} \equiv (V_{1},E_{1})$ a subgraaph of $G$ and $G_{1}^{c}$ its complement. With respect to $G$ then $\mu_{G}^{E}(v_{i}) + \mu_{G'}^{E}(v_{i}) = 1$ for all $v_{i} \in V(G)$

**Proof**

From the definition of membership and $\frac{|:\{v \in V(G) \mid v \notin G_{1}\}|}{|V(G)|} = 1$

**Proposition 3.3**

Let $G \equiv (V,E)$ be a simple connected graph and $G_{1} \equiv (V_{1},E_{1}), G_{2} \equiv (V_{2},E_{2})$ are subgraphs of $G$ and $G_{2} \subseteq G_{1}$

1. $\mu_{G_{1}}^{E}(v_{i}) = 1$ iff $v_{i} \in L_{G_{1}}(V_{2})$
2. $\mu_{G_{1}}^{E}(v_{i}) = 0$ iff $v_{i} \in (V - U_{G_{1}}(V_{2}))$
3. $0 \leq \mu_{G_{1}}^{E}(v_{i}) \leq 1$ iff $v_{i} \in (U_{G_{1}}(V_{2}) - L_{G_{1}}(V_{2}))$

**Proof**

obviously

From above we can fuzzify a subgraph of a simple connected graph, by using membership function as characteristic function of all vertices of a subgraph, we can obtain a fuzzy set of graph vertices as $F_{G_{1}}^{E}(v_{i}) = \{(v, \mu_{G_{1}}^{E}(v_{i})) \mid \text{for every } G_{1} \subseteq G \text{ and } v_{i} \in V(G)\}$.

**Proposition 3.4**

Let $G \equiv (V,E)$ be a simple connected graph and $G_{1} \equiv (V_{1},E_{1}), G_{2} \equiv (V_{2},E_{2})$ are subgraphs

(i) $F_{G_{1} \cup G_{2}}^{E} \supseteq F_{G_{1}}^{E} \cup F_{G_{2}}^{E}$

(ii) $F_{G_{1} \cup G_{2}}^{E} = F_{G_{1}}^{E} \cup F_{G_{2}}^{E}$ if either $G_{1} \subseteq G_{2}$ or $G_{2} \subseteq G_{1}$

**Proof**

(i) From $\mu_{G_{1} \cup G_{2}}^{E}(v_{i}) \geq \max_{v_{i} \in V}(\mu_{G_{1}}^{E}(v_{i}), \mu_{G_{2}}^{E}(v_{i}))$

Pawlak forms of rough set theory applied on a special cases of graph theory in [9,18] Thus, our new work may be Redefined another forms in the sense of precision coefficient k (k$\in (0, 1])$

**Definition 3.1**

Let $G \equiv (V,E)$ be a simple connected graph and $G_{1} \equiv (V_{1},E_{1})$ subgraphs of $G$. k-lower and k-upper approximation of $X$ based on rough membership function may be defined as

$L_{G_{1}}^{k}(X) = \{v_{i} \in X: \mu_{G_{1}}^{E}(v_{i}) \geq k, X \subseteq V_{1}\}$

$U_{G_{1}}^{k}(X) = \{v_{i} \in X: \mu_{G_{1}}^{E}(v_{i}) \geq 1 - k, X \subseteq V_{1}\}$

From above $X$ is k-exact if and only if $L_{G_{1}}^{k}(X) = U_{G_{1}}^{k}(X)$ Otherwise $X$ is k-rough. We try to obtain some properties between the new definitions by comparing rough sub graphs model with each others

**Theorem 3.1**

Let $G \equiv (V,E)$ be a simple connected graph, $G_{1} \equiv (V_{1},E_{1})$, subgraphs of $G$ and $X \subseteq Y \subseteq V$

1) $L_{G_{1}}^{k}(\emptyset) = \emptyset$, $U_{G_{1}}^{k}(\emptyset) = \emptyset$
2) $L_{G_{1}}^{k}(V) = V$, $U_{G_{1}}^{k}(V) = V$
3) $L_{G_{1}}^{k}(L_{G_{1}}^{k}(X)) = V - (U_{G_{1}}^{k}(V - X))$<br>4) $U_{G_{1}}^{k}(U_{G_{1}}^{k}(X)) = V - (L_{G_{1}}^{k}(V - X))$
5) $L_{G_{1}}^{k}(X \cup Y) \supseteq L_{G_{1}}^{k}(X) \cup L_{G_{1}}^{k}(Y)$
6) $U_{G_{1}}^{k}(X \cup Y) \supseteq U_{G_{1}}^{k}(X) \cup U_{G_{1}}^{k}(Y)$
Example 3.1

From Example 2.2 $\mu_{G_1}^k(a) = 1$, $\mu_{G_1}^k(b) = \frac{2}{3}$, $\mu_{G_2}^k(c) = 1$, $\mu_{G_2}^k(d) = 1$. Consider $X = \{a,b\}, Y = \{a\}$.

$U_{G_1}^{0.5}(X) \subseteq U_{G_1}^{0.5}(X)$ and $U_{G_1}^{0.5}(X) \subseteq U_{G_1}^{0.5}(X), L_{G_1}^{0.5}(Y) \subseteq L_{G_1}^{0.5}(X)$ and $U_{G_1}^{0.5}(Y) \subseteq U_{G_1}^{0.5}(X)$.
IV. Conclusion

Rough set theory has been regarded as tool to deal with the uncertainty or imprecision information, the graded rough set model based on two distinct but related universes was proposed. But it is still restrictive for many applications, the rough membership function based type graded rough set in any graph which based on covering neighborhood of all vertices. We have some interesting properties and conclusions about rough set model on many sub graphs and paths, which can help us understand the approximations structure of graphs. In the future we will further study other types of operations on graphs such that deletions vertices or edges by using rough sets and so on.

References