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On Conjugate EP Matrices in Indefinite Inner Product Spaces

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Abstract: *The aim of this article is to introduce the concept of Conjugate EP(Con-J-EP) matrices in the setting of indefinite inner product spaces with respect to the indefinite matrix product. Relation between Con-J-EP and Con-EP matrices are discussed.*
Keywords: *Indefinite matrix product, indefinite inner product space, EP matrices, Con-EP matrices, J-EP matrices.*

I. Introduction

An indefinite inner product in $\mathbb{C}^n$ is a conjugate symmetric sesquilinear form [x, y] together with the regularity condition that [x, y] = 0, ∀ y ϵ $\mathbb{C}^n$ only when x = 0. Any indefinite inner product is associated with a unique invertible Hermitian matrix J with complex entries such that [x, y] = < x, Jy >, where <. , . > denotes the Euclidean inner product on $\mathbb{C}^n$, with an additional assumption on J, that is, $J^2 = I$, to present the results with much algebraic ease. Thus an indefinite inner product space is a generalization of Minkowski space, where the weight is a diagonal matrix of order n, with the first entry 1 and the remaining entries are all -1.

Investigations of linear maps on indefinite inner product spaces employ the usual multiplication of matrices which is induced by the Euclidean inner product of vectors (see for instance [2, 4] and the references cited therein). This causes a problem as there are two different values for dot product of vectors. To overcome this difficulty, Kamaraj, Ramanathan and Sivakumar introduced a new matrix product called indefinite matrix multiplication and investigated some of its properties in [4]. More precisely, the indefinite matrix product of two matrices A and B of sizes $m \times n$ and $n \times l$ complex matrices, respectively, is defined to be the matrix $A \circ B = AJ_nB$. The adjoint of A, denoted by $A^{[*]}$ is defined to be the matrix $J_nA^*J_m$, where J_n and J_m are weights in the appropriate spaces. With this new matrix product S. Jayaraman and Ivana M.Radojević, Niš introduced and studied J-EP matrices as a generalization of EP matrices in indefinite inner product spaces in [3] and [5] respectively.

In this paper we introduce the concept of Conjugate EP matrices in the setting of indefinite inner product spaces with respect to the indefinite matrix product. Relation between Con-J-EP and Con-EP matrices are discussed.

II. Notations, Definitions and Preliminaries

We first recall the notion of an indefinite multiplication of matrices. We refer the reader to [4], wherein various properties and also advantages of this product have been discussed in detail.

Definition 2.1. Let A and B be $m \times n$ and $n \times l$ complex matrices, respectively. Let J_n be an arbitrary but fixed $n \times n$ complex matrix such that $J_n = {J_n}^* = {J_n}^{-1}$. The indefinite matrix product of A and B (relative to J_n) is defined by $A \circ B = AJ_nB$.

Definition 2.2 Let A be an $m \times n$ complex matrix. The adjoint $A^{[*]}$ of A is defined by $A^{[*]} = J_nA^*J_m$.

Definition 2.3 For $A \in \mathbb{C}^{m\times n}$, a matrix $X \in \mathbb{C}^{n\times m}$ is called the Moore-Penrose inverse of A if it satisfies the following equations: $A \circ X \circ A = A, X \circ A \circ X = X$, $(A \circ X)^{[*]} = A \circ X$ and $(X \circ A)^{[*]} = X \circ A$.

Definition 2.4 For $A \in \mathbb{C}^{m\times n}$, a matrix $X \in \mathbb{C}^{n\times m}$ is called the group inverse of A if it satisfies the following equations: $A \circ X \circ A = A$, $X \circ A \circ X = X$, $A \circ X = X \circ A$.

Remark 2.5 We are familiar with the fact that for $A \in \mathbb{C}^{m\times n}$ the Moore- Penrose inverse has the form $A^{[\dagger]} = J_nA^{\dagger}J_m$, and it always exists because the condition rank($A^{[*]} \circ A$) = rank($A \circ A^{[*]}$) = rank(A) is always satisfied. On the other hand, it is not the case that a similar formula for the group inverse holds. It may happen that the group inverse in the Euclidean space exists, but in the space with indefinite matrix product it does not, and vice versa, for example, for $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Anyway, $A^{[\#]} = (AJ)^{\#}J$, and it exists if and only if $\mathrm{rank}(A^{(2)}) = \mathrm{rank}(A)$, ie., $\mathrm{rank}(AJA) = \mathrm{rank}(A)$, while $A^{\#}$ exists if and only if $\mathrm{rank}(A^2) = \mathrm{rank}(A)$. Clearly, if A and J commute, then both the group inverses exist at the same time and, in that case $A^{[\#]} = A^{\#}$.

Definition 2.6 [3] A matrix $A \in \mathbb{C}^{n\times n}$ is J-EP if $A \circ A^{[\dagger]} = A^{[\dagger]} \circ A$.

Lemma 2.7 [3] A matrix $A \in \mathbb{C}^{n\times n}$ is J-EP if and only if AJ is an EP matrix.

Definition 2.8 [6] A matrix $A \in \mathbb{C}^{n\times n}$ is said to be Con-EP if $A A^{\dagger} = \overline{A^{\dagger}A}$.

III. Conjugate J-EP Matrices

Definition 3.1 A matrix $A \in \mathbb{C}^{n\times n}$ is said to be Con-J-EP if $A \circ A^{[\dagger]} = \overline{A^{[\dagger]} \circ A}$.

Remark 3.2 The definition of Con-J-EP matrices coincides with that of Con-EP, when $J = I$. However Con-EP matrices need not be Con-J-EP and vice versa.

Example 3.3 Let $A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$, $J = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$.
Clearly A is Con-EP.

$$A^{\dagger} = \frac{1}{4}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}; \qquad A^{[\dagger]} = \frac{1}{2}\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}$$

$$A \circ A^{[\dagger]} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & i \\ 0 & -1 \end{pmatrix}; \qquad \overline{A^{[\dagger]} \circ A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}$$

Therefore $A \circ A^{[\dagger]} \neq \overline{A^{[\dagger]} \circ A}$.
Hence A is not Con J-EP.

Example 3.4 Let $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$, $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
Then $A^{\dagger} = \frac{1}{4}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ and $A^{[\dagger]} = \frac{1}{4}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$ and
A is Con-J-EP but not Con-EP.

Theorem 3.5: Let $A, B \in \mathbb{C}^{n\times n}$ and $J \in R^{n\times n}$. Then we have

(i) AJ is Con-EP if and only if A is con-J-EP
(ii) If J commutes with A, then A is Con-EP if and only if A is Con-J-EP.
(iii) If J commutes with A , then AB is Con-EP if and only if $A \circ B$ is Con-J-EP

Proof:

(i) AJ is Con-EP $\Leftrightarrow (AJ)(AJ)^{\dagger} = \overline{(AJ)^{\dagger}(AJ)}$

$$\begin{aligned}
&\Leftrightarrow AJJA^{\dagger} = \overline{JA^{\dagger}AJ}\\
&\Leftrightarrow AA^{\dagger} = \overline{JA^{\dagger}AJ}\\
&\Leftrightarrow AA^{\dagger}J = \overline{JA^{\dagger}A}\\
&\Leftrightarrow AJJA^{\dagger}J = \overline{JA^{\dagger}JJA}\\
&\Leftrightarrow A \circ A^{[\dagger]} = \overline{A^{[\dagger]} \circ A}\\
&\Leftrightarrow A \text{ is Con-J-EP.}
\end{aligned}$$

(ii) A is Con-J-EP $\Leftrightarrow A \circ A^{[\dagger]} = \overline{A^{[\dagger]} \circ A}$

$$\begin{aligned}
&\Leftrightarrow AA^{\dagger}J = \overline{JA^{\dagger}A}\\
&\Leftrightarrow AA^{\dagger} = \overline{JA^{\dagger}AJ}\\
&\Leftrightarrow AA^{\dagger} = \overline{(AJ)^{\dagger}(JA)}\\
&\Leftrightarrow AA^{\dagger} = \overline{(JA)^{\dagger}(JA)}\\
&\Leftrightarrow AA^{\dagger} = \overline{A^{\dagger}JJA}\\
&\Leftrightarrow AA^{\dagger} = \overline{A^{\dagger}A}\\
&\Leftrightarrow A \text{ is Con-EP.}
\end{aligned}$$

(iii) Suppose $AJ = JA$. Then,

$$A \circ B \text{ is Con-J-EP} \Leftrightarrow (A \circ B) \circ (A \circ B)^{[\dagger]} = \overline{(A \circ B)^{[\dagger]} \circ (A \circ B)}$$
$$\Leftrightarrow (AJB)JJ(AJB)^{\dagger}J = \overline{J(AJB)^{\dagger}JJAJB}$$

$$\Leftrightarrow \mathrm{JABB^{\dagger}J^{\dagger}A^{\dagger}J} = \overline{\mathrm{JB^{\dagger}J^{\dagger}A^{\dagger}JAB}}$$
$$\Leftrightarrow \mathrm{JABB^{\dagger}A^{\dagger}} = \overline{\mathrm{JB^{\dagger}A^{\dagger}AB}}$$
$$\Leftrightarrow \mathrm{J(AB)(AB)^{\dagger}} = \overline{\mathrm{J(AB)^{\dagger}AB}}$$
$$\Leftrightarrow \mathrm{(AB)(AB)^{\dagger}} = \overline{\mathrm{(AB)^{\dagger}AB}}$$
$\Leftrightarrow$ AB is Con-EP. ■

The following examples show that the commutativity assumption in the above theorem cannot be dispensed with.

Example 3.6 Let $A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$, $J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$A^{\dagger} = \frac{1}{4}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

Clearly A is Con-EP.

However, A is not Con-J-EP as $A^{[\dagger]} = \frac{1}{4}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ and

$$A \circ A^{[\dagger]} = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$$

While

$$\overline{A^{[\dagger]} \circ A} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

Note that $AJ \neq JA$.

Example 3.7 Let $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ and $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $A^{\dagger} = \frac{1}{4}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$.
$A^{[\dagger]} = \frac{1}{4}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$ and A is Con-J-EP but not Con-EP.
Again note that, in this case also, $AJ \neq JA$.

Example 3.8 Let $A = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
Then $AB = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ which is Con-EP.
However, $A \circ B = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$ which is not Con-J-EP.
Here we note that $AJ \neq JA$.

Example 3.9 Let $A = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
Then $AB = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ which is not Con-EP.
However, $A \circ B = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ which is Con-J-EP.
Here we note that $AJ \neq JA$.

Theorem 3.10 Let $A \in \mathbb{C}^{n\times n}$, $J \in R^{n\times n}$ and J commute with $AA^{\dagger}$.
Then A is Con-J-EP if and only if A is Con-EP.
Proof: Let J commute with $AA^{\dagger}$ and let A be a Con-J-EP matrix. Then

$$AA^{\dagger} = AA^{\dagger}JJ = JAA^{\dagger}J = JAJJA^{\dagger}J = JA \circ A^{[\dagger]} = J\overline{A^{[\dagger]} \circ A} = \overline{A^{\dagger}A}.$$

Thus A is Con-EP matrix.
Conversely, let J commute with $A A^{\dagger}$ and let A be a Con-EP matrix.
Now, we have

$$A \circ A^{[\dagger]} = AJJA^{\dagger}J = AA^{\dagger}J = JAA^{\dagger} = \overline{JA^{\dagger}A} = \overline{JA^{\dagger}JJA} = \overline{A^{[\dagger]} \circ A}.$$

Thus A is a Con-J-EP matrix. ■
Also we have an analogous theorem.

Theorem 3.11 Let $A \in \mathbb{C}^{n\times n}$, $J \in R^{n\times n}$ and J commute with $A^{\dagger}A$.
Then A is Con-J-EP if and only if A is Con-EP.

III. Conclusion

Investigation into group inverse of Con-J-EP matrices and sums of Con-J-EP matrices with respect to indefinite matrix multiplication is currently being undertaken.

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