A Note on $K_r$ Excellent Domination Parameter

N. Venkataraman
Assistant professor The department of mathematics Gac Ooty 14.4.2015

Abstract: Let $G = (V, E)$ be a simple graph of order $p$ and size $q$. A subset $S$ of $V$ is said to be a $K_r$-dominating set of $G$ if for every vertex $v \in (V - S)$ is $K_r$-adjacent to at least one vertex in $S$. Since $v$ is always a $K_r$-dominating set, for every $r$, the existence of $K_r$-dominating set in $G$ is guaranteed. A $K_r$-dominating set of minimum cardinality is called a minimum $K_r$-dominating set and its cardinality is denoted by $\gamma_{K_r}$. Clearly $\gamma = \gamma_{K_2}$ and $\gamma \leq \gamma_{K_r}$ for every $r > 2$.

I. Introduction

A vertex $v$ is said to be $k_r$-adjacent to a vertex $u$ if $u$ and $v$ are contained in a $r$-clique of $G$. Let $u \in V(G)$, define $k_r$-neighbourhood, denoted by $N_{k_r}(v) = \{v \in V : v k_r-adjacent to u\}$. If $N_{k_r}(u) = \emptyset$ then $u$ is called a $k_r$-isolated vertex. Let $G = (V, E)$ be a graph. A subset $S \subseteq V$ is said to be a $K_r$-dominating set of $G$ if for every vertex $v \in (V - S)$ is $K_r$-adjacent to at least one vertex in $S$. A tree $T$ is said to be $\gamma_{K_r}$-excellent if for every vertex of $T$ is some $\gamma_{K_r}$-set.

Results: For any $n$, if $G$ does not contain any $r$-clique, then $\gamma_{K_r} = p$. In particular, if $r > p$ then $\gamma_{K_r} = p$. Therefore we assume that $r \leq p$. 

(i) $\gamma_{K_r}(K_r) = 1$
(ii) $\gamma_{K_r}(S_1, p) = \begin{cases} 1 & \text{if } r = 2 \\ (p + 1) & \text{if } r > 2 \end{cases}$
(iii) $\gamma_{K_r}(W_n) = \begin{cases} (\frac{p(p-3k)}{2}) & \text{if } r = 3 \\ (p + 1) & \text{if } r \geq 4 \end{cases}$
(iv) $\gamma_{K_r}(P_n) = \begin{cases} (p) & \text{if } r > 2 \\ \emptyset & \text{otherwise} \end{cases}$
(v) $\gamma_{K_r}(G) = \begin{cases} (p) & \text{if } r = 2 \\ (p + 1) & \text{if } r > 2 \end{cases}$
(vii) $\gamma_{K_r}(K_{p_1,p_2}) = \begin{cases} 2 & \text{if } r = 2 \\ (p_1 + p_2) & \text{Otherwise} \end{cases}$
(viii) If $r > 2$ then any $\gamma_{K_r}$-set contains all pendant vertices. Therefore $\gamma_{K_r}(G \circ K_1) = (p + \gamma_{K_r}(G))$ if $r > 2$.

Ore’s Theorem

II Statement:

A $K_r$-dominating set $S$ of a graph $G$ is minimal if and only if for every $u \in S$ either or both of the following conditions hold.
(i) $N_{k_r}(u) \cap S = \emptyset$
(ii) $\exists$ a vertex $v \in (V - S)$ such that $N_{k_r}(u) \cap S = u$.

Proof: Let $S$ be a $K_r$-dominating set. Then obviously any $u \in S$. Conditions (i) (or) (ii) (or) both. Conversely, assume that for every $u \in S$, conditions (i) (or) (ii) (or) both holds.

Claim: $S$ is a minimal $K_r$-dominating set.
Suppose not. Then there exists \( u \in S \) such that \( (S - u) \) is a \( K_r \)-dominating set. That is, there exists \( v \in S \) such that \( u \) is \( K_r \)-adjacent to \( v \). (i.e) \( N_u \cap S \neq \varnothing \). (i.e) (i) is not satisfied. Therefore \( u \) satisfies condition(ii). (i.e) there exists \( v \in (V - S) \) such that \( N_v(v) \cap (S - u) \neq \varnothing \). Therefore \( (S - u) \) is not a \( K_r \)-dominating. Which is a contradiction. Hence the thm.

**Remark:**
Let \( G = (V, E) \) be a graph with a vertices. Let \( r \geq 2 \). Then \( 1 \leq \gamma_k \leq n \) and these bounds are Sharp.

**Remark:**
Any \( K_r \) dominating set with \( r \geq 3 \) contains all pendent vertices. Also, for a tree \( T \), \( V(T) \) is the minimum \( K_r \) dominating set for all \( r \geq 3 \).

**iii Result**
A graph \( G \) has \( V \), as its unique \( K_r \) dominating set if and if \( G \) contains no \( r \) \(-\) clique.

**Proof**
\[ \Rightarrow \]
Assume that \( G \) contains no \( r \)-clique. To prove that \( G \) has \( V \) as its unique \( K_r \)-dominating set. The proof is Obvious.

\[ \Leftarrow \]
Assume that graph \( G \) has \( V \) as its unique \( K_r \) dominating set. To prove that \( G \) contains no \( r \)-clique. Suppose \( G \) contains a \( r \)-clique say \( \{v_1, v_2, ..., v_r\} \). Then \( (V - \{v_2, v_3, ..., v_r\}) \) is a \( K_r \)-dominating set. Which is a contradiction to our hypothesis.

**Corollary**
If their exists \( v \in V \) such that \( N[V] \) contains a \( r \)-clique. Then \( \gamma_k (G) \leq n - r + 1 \).

**Observation**
A graph \( G \) has \( \gamma_k (G) = 1 \) if and only if there exists a point \( u \in V(G) \) such that every point is in a \( r \)-clique containing \( u \). (i.e) if and only if \( G \) is \( K_n \) with \( r \geq n \) or \( G \) is obtained from union of cliques, each of size \( \geq n - 1 \) and joining every point of each clique to a new point.

**Result**
Let \( S \) be a \( \gamma \)-set. Let the number of points in \( (V - S) \) which are not \( K_r \)-adjacent to any of the vertices of \( S \) be \( t \). Then \( \gamma_k (G) = \gamma(G) + t \).

iv **Examples:**

\[ \gamma = 1 = \gamma_k \gamma_k = 1 + 1 = 2 \]

**Theorem**
**Statement**
Every graph of order \( p \) is an induced subgraph of a \( \gamma_k \)-excellent graph.

**Proof**
Let \( G \) be a graph of order \( p \). Attach at each point \( v \), a complete graph \( K_{r-1} \) with \( v \) as one of the vertices. The resulting graph is denoted by \( GoK_r \). The graph \( G \) is an induced subgraph of \( GoK_r \) which is \( \gamma_k \)-excellent. Hence the theorem.

**Corollary:**
There does not exists a forbidden sub graph characterization of the class of \( v_{\gamma_{k_2}} \)-excellent graphs.

Examples:

Subdivided graph (or) Star is not \( K_2 \)-excellent.

\( C_4 \) is \( K_2 \)-excellent but not \( K_3 \)-excellent.

\( K_3 \) is \( K_3 \)-excellent.

\( K_n \) is not \( K_r \)-excellent.

**Note:**
A tree \( T \) is \( \gamma_{k_2} \)-excellent, for all \( n \geq 3 \), \( \gamma_{k_2} \)-excellent, tree has already been characterized by sumner.

**Definition:**
A connected graph \( G \) is called a \( K_r \)-tree if every vertex is in a \( K_r \)-clique and \( G \) does not contain \( C_m \) where \( m \geq r + 1 \).

**Remark:**
A K_r-tree is simply a tree.

Example:

\[ v_4, v_3 \text{ K}_3 \text{-tree.} \]

**Definition:**
A pendant vertex \( v \) of a K_r-tree of a graph \( G \) is a vertex which contain exactly one K_r-clique.

**Note:**
For any \( r \), if \( G \) does not contain any \( r \)-clique, then \( \gamma_{kr} = p \). In particular if \( r > p \) then \( \gamma_{kr} = p \). Therefore we assume that \( r \leq p \).

(i) \( \gamma_{kr}(K_p) = 1 \)

(ii) \( \gamma_{kr}(P_p) = \frac{d(p/3)e}{r-2} \) if \( r = 2 \)

\( \gamma_{kr}(P_p) = p \) if \( r > 2 \). Where \( P \) is Peterson’s graph.

**Definition:**
A K_r-path is a K_r-tree containing exactly K_r-pendant vertices and every other point is contained in exactly two K_r-cliques.

**Note:**
A K_r-path is a K_3-Tree in which there are exactly three K_3-pendant vertices.

Examples:

Remark:
(i) \( d_{kr}(v) = K_r \)-degree of \( v \) = number of \( r \)-cliques containing \( v \).

(ii) Length of a K_r-path \( P \) denoted by \( l(P) \) is the number of K_r’s (r-cliques) present in the path.

Theorem:
A $k_r$-path $P$ is $\gamma_{k_r}$-excellent $\iff l(P) = 1 \text{(or)} l(P) = 0 \text{(mod 3)}$.

**Uses:**
$K_r$-domination has application in communication network system for rapid transfer of shared information among the members of the core group.

**References**

 "A note on packing two trees into $K_r$,”
[4]. "Fundamentals of domination in graphs”