Common Fixed Point Theorems for Four Weakly Compatible Self-Mappings in Fuzzy Metric Space Using (JCLR) Property

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Abstract: In this paper we prove a common fixed point theorem for four weakly compatible self-mappings in fuzzy metric space by using (JCLR) property. An example is given which shows the validity of main theorem. We also extend the result to two finite families of self-mappings with pairwise commuting.

Key Words: Fuzzy Metric Space, Weakly Compatible Mappings, E.A Property, (CLR) Property and (JCLR) Property.

I. Introduction

In 1965, Zadeh [33] introduced the concept of fuzzy set. In the last two decades tremendous development has taken place in the study of fuzzy set theory. Fuzzy set theory has applications in applied sciences such as neural network, stability theory, mathematical programming, modeling theory, image processing, control theory, communication, engineering sciences, medical sciences etc.


In 2002, Aamri and El-Moutawakil [1] defined the notion of E.A property for self-mappings which contains the class of non-compatible mappings in metric spaces. E.A property replaces the completeness condition by closedness condition of the range and relaxes the completeness of the whole space, continuity of one or more mappings and containment of the range of one mapping into the range of the other.

A number theorems were proved for self-mappings satisfying E.A property [or CLR’s property]. Many authors have proved common fixed point theorems in fuzzy metric spaces for different contractive conditions. For details refer to [3, 5-16, 17, 20, 22, 24, 25, 26, 27, 28, 31, 32].

II. Preliminaries

2.1 Fuzzy Metric Space

A fuzzy metric space is a triple $(X, M, T)$ where $X$ is a nonempty set, $T$ is a continuous $t$-norm and $M$ is a fuzzy set on $X \times X \times (0, \infty)$ and the following conditions are satisfied for all $x, y \in X$ and $t, s > 0$:

(FM-1) $M(x, y, t) > 0$;
(FM-2) $M(x, y, t) = 1 \iff x = y$;
(FM-3) $M(x, y, t) = M(y, x, t)$;
(FM-4) $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous;
(FM-5) $M(x, z, t + s) \geq T(M(x, y, t), M(y, z, s))$. □

2.2 Weakly Compatible Mappings

Two self-mappings $f$ and $g$ of a non-empty set $X$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e., if $fx = gz$ for some $z \in X$, then $gfz = gfz$. □

2.3 E.A Property

Two self-mappings $f$ and $g$ of a fuzzy metric space $(X, M, T)$ are said to satisfy E.A. property, if there exists a sequence $(x_n)$ in $X$ such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = u$ for some $u \in X$. □

2.4 (CLR) Property

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Let $f$ and $g$ be self-mappings on a fuzzy metric space $(X, M, T)$. Then the pair $(f, g)$ is said to satisfy CLRg property (common limit in the range for any coordinate and for all $n$) if
\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = gx
\]
for some $x \in X$. □

2.5 Increasing Function
Let $\Phi$ be class of all mappings $\phi : [0, 1] \to [0, 1]$ satisfying the following conditions:

(φ1) $\phi$ is continuous and nondecreasing on $[0, 1]$;

(φ2) $\phi(x) > x$ for all $x \in (0, 1)$.

2.6 Lemma
Let $(X, M, T)$ be a fuzzy metric space. If there exists $k \in (0, 1)$ such that
\[
M(x, y, kt) \geq M(x, y, t) \quad \text{for all } x, y \in X \text{ and } t > 0,
\]
then $x = y$. □

III. Main Result
In this paper, we first introduce the notion of the joint common limit in the range property of two pairs of self-mappings.

3.1 Definition
Let $(X, M, T)$ be a fuzzy metric space and $f, g, p, q : X \to X$ be self-mappings. The pairs $(f, q)$ and $(p, g)$ are said to satisfy the joint common limit in the range of $q$ and $g$ property (shortly, $(JCLRg)$ property) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that
\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} qx_n = \lim_{n \to \infty} py_n = \lim_{n \to \infty} gy_n = qy = gy,
\]
for some $u \in X$.

3.2 Remark
If $p = f, q = g$ and $\{x_n\} = \{y_n\}$ in (1), then we get the definition of $(CLRg)$ property. □

Throughout this section, $\Phi$ denotes the set of all continuous and increasing functions $\varphi : [0, 1]^4 \to [0, 1]$ in any coordinate and $\varphi(t, t, t, t) > t$ for all $t \in [0, 1]$.

Following are examples of some functions $\varphi \in \Phi$:

1) $\varphi(x_1, x_2, x_3, x_4, x_5) = (\min \{x_i\})^h$, for some $0 < h < 1$.
2) $\varphi(x_1, x_2, x_3, x_4, x_5) = x_i^h$ for some $0 < h < 1$.
3) $\varphi(x_1, x_2, x_3, x_4, x_5) = T(T(T(x_1, x_2), x_3), x_4, x_5)^h$, for some $0 < h < 1$ and for all $t$-norms $T$ such that $T(t, t) = t$.

Now, we state and prove our main result.

3.3 Theorem
Let $(X, M, T)$ be a fuzzy metric space and $f, g, p$ and $q$ be mappings from $X$ into itself. Further, let the pairs $(f, q)$ and $(p, g)$ are weakly compatible and there exists a constant $k \in 0, \frac{1}{2}$ such that
\[
(qx, gy, t), M(fx, qx, t), M(py, gy, t),
\]

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\[ M(\, fx, py, kt) \succeq \varphi \] -- (2)

\[ M(fx, gy, at), M(py, qx, 2t-αt) \]

holds for all \( x, y \in X, \alpha \in (0, 2), \ t > 0 \) and \( \varphi \in \Phi. \]

If \( (f, q) \) and \( (p, g) \)

\( M(\, fx, py, at) M(\, py, qx, 2t-αt) \), \( M(\, py, qx, 2t-αt) \)

satisfy the (JCLRqg) property, then \( f, g, p \) and \( q \)

have a unique common fixed point in \( X. \)

**Proof:**

Since the pairs \( (f, q) \) and \( (p, g) \) satisfy the (JCLRqg) property, there exist sequences \( \{x_n\} \) and \( \{y_n\} \) in \( X \) such that

\[
\lim_{n \to \infty} fx = \lim_{n \to \infty} qx = \lim_{n \to \infty} py = \lim_{n \to \infty} gy = qu = gu,
\]

for some \( u \in X. \)

Now, we assert that \( gu = pu. \) Using (2), with \( x = x_n, y = u, \ \alpha = 1, \)

we get

\[
M(\, qx_n, gu, t), M(\, fx_n, qx_n, t), M(\, pu, gu, t),
\]

\[
M(\, fx, pu, kt) \succeq \varphi
\]

Taking the limit as \( n \to \infty, \) we have

\[
M(\, gu, gu, t), M(\, gu, gu, t), M(\, pu, gu, t),
\]

\[
M(\, gu, pu, kt) \succeq \varphi
\]

\[
M(\, gu, gu, t), M(\, pu, gu, t)
\]

Since \( \varphi \) is increasing in each of its coordinate and \( \varphi(t, t, t, t, t) > t \) for all \( t \in [0, 1), \) we get \( M(\, gu, pu, kt) \succeq M(\, gu, pu, t). \) By Lemma 2.6, we get \( gu = pu. \)

Next, we show that \( fu = gu. \) Using (2), with \( x = u, y = y_n, \ \alpha = 1, \)

we get

\[
M(\, fu, gy_n, kt) \succeq M(\, gu, gy_n, t), M(\, pu, gu, t)
\]

Taking the limit as \( n \to \infty, \) we have

\[
M(\, fu, gu, t), M(\, fu, gu, t), M(\, fu, gu, t), M(\, fu, gu, kt) \]

\[
M(\, fu, gu, t) \succeq \varphi
\]

Since \( \varphi \) is increasing in each of its coordinate and \( \varphi(t, t, t, t) \in [0, 1), \) we get \( M(\, fu, gu, kt) \succeq M(\, fu, gu, t). \) By Lemma 2.6, we have \( fu = gu. \)

\[
(\, gu, gu, t) \succeq M(\, gu, gu, t)
\]

\[
(\, gu, gu, t)
\]

\[
t, t, t > t \text{ for all}
\]
Now, we assume that \( z = fu = gu = pu = qu \). Since the pair \((f, q)\) is weakly compatible, \( fqu = qfu \) and then \( fz = fqu = qfu = qz \). Again since \((p, g)\) is weakly compatible, it follows \( gpu = pgu \) and hence \( gz = gpu = pgu = pz \).

We show that \( z = fz \). To prove this, using (2), with \( x = z, y = u, \alpha = 1 \), we get
\[
M(fz, pu, kt) \geq \varphi
\]
and so
\[
M(fz, gu, t), M(fz, qz, t), M(pu, gu, t), M(fz, pu, kt) \geq \varphi
\]
\[
M(fz, gu, t), M(pu, qz, t)
\]
\[
M(fz, z, t), M(fz, fz, t), M(z, z, t)
\]
\[
M(fz, z, kt) \geq \varphi
\]
\[
M(fz, z, t), M(z, fz, t)
\]
Since \( \varphi \) is increasing in each of its coordinate and \( \varphi(t, t, t, t, t, t) > t \) for all \( t \in [0, 1] \), \( M(fz, z, kt) \geq M(fz, z, t) \), which implies that \( fz = z \). Hence \( z = fz = qz \).

Next, we show that \( z = pz \). To prove this, using (2), with \( x = u, y = z, \alpha = 1 \), we get
\[
M(fu, pz, kt) \geq \varphi
\]
and so
\[
M(fu, gz, t), M(fu, qu, t), M(pz, gz, t), M(fz, gu, t), M(pu, qz, t)
\]
\[
M(fz, gz, t), M(pz, qu, t)
\]
\[
M(z, pz, t), M(z, z, t), M(pz, pz, t), M(z, pz, kt) \geq \varphi
\]
\[
M(z, pz, t), M(pz, z, t)
\]
Since \( \varphi \) is increasing in each of its coordinate and \( \varphi(t, t, t, t, t, t) > t \) for all \( t \in [0, 1] \), \( M(z, pz, kt) \geq M(pz, pz, t) \), which implies that \( z = pz \).

Hence \( z = pz = gz \). Therefore, we conclude that \( z = fz = gz = pz = qz \).

This implies \( f, g, p \) and \( q \) have a common fixed point \( z \).

For uniqueness of common fixed point, we let \( w \) be another common fixed point of the mappings \( f, g, p \) and \( q \). On using (2) with \( x = z, y = w, \alpha = 1 \), we have
\[
M(qz, gw, t), M(fz, qz, t), M(pw, gw, t), M(fz, pw, kt) \geq \varphi
\]
and then
\[
M(fz, gw, t), M(pw, qz, t), M(fz, gw, t), M(pw, qz, t)
\]
\[
M(z, w, t), M(z, z, t), M(w, w, t), M(z, w, kt) \geq \varphi
\]
\[
M(z, w, t), M(w, z, t)
\]
Since \( \varphi \) is increasing in each of its coordinate and \( \varphi (t, t, t) > t \) for all \( t \in [0, 1) \), \( M(z, w, kt) \geq M(z, w, t) \), which implies that \( z = w \).

Therefore \( f, g, p \) and \( q \) have a unique common fixed point. \( \square \)

3.4 **Remark**

From the results, it is asserted that (JCLRqg) property never requires conditions closedness of the subspace, continuity of one or more mappings and containment of ranges amongst involved mappings.

3.5 **Remark**

Theorem 3.3 improves and generalizes the results of Abbas et al. ([2], Theorem 2.1) and Kumar ([19], Theorem 2.3) without any requirement of containment amongst range sets of the involved mappings and closedness of the underlying subspace.

3.6 **Remark**

Since the condition of \( t \)-norm with \( T(t, t) = t \) for all \( t \in [0, 1] \) is replaced by arbitrary continuous \( t \)-norm, Theorem 3.3 also improves the result of Cho et al. ([4], Theorem 3.1) without any requirement of completeness of the whole space, continuity of one or more mappings and containment of ranges amongst involved mappings.

3.7 **Corollary**

Let \((X, M, T)\) be a fuzzy metric space and \( f, g, p \) and \( q \) be mappings from \( X \) into itself. Further, let the pairs \((f, q)\) and \((p, g)\) are weakly compatible and there exists a constant \( k \in (0, 1] \) such that

\[
M(fx, py, kt) \geq a(t)M(qx, gy, t) + a(t)M(py, gy, 2t - at) + a(t)M(qx, gy, 2) + a(t)M(py, gy, 2) + a(t)M(qx, gy, 2)
\]

holds for all \( x, y \in X, \alpha \in (0, 2), t > 0 \) and \( a : \alpha \rightarrow (0, 1] \) such that \( \sum_{i=1}^{5} a_i(t) = 1 \). If \( (f, q) \) and \((p, g)\) satisfy the (JCLRqg) property, then \( f, g, p \) and \( q \) have a unique common fixed point in \( X \).

**Proof:**

By Theorem 3.3, we define

\[
\varphi (x, x, x, x, x) = a(t)x + a(t)x + a(t)x + a(t)x + a(t)x + a(t)x + a(t)x + a(t)x + a(t)x + a(t)x
\]

then the result follows.

3.8 **Remark**

Corollary 3.7 improves the result of Cho et al. ([4], Theorem 3.1) without any requirement of completeness of the whole space, continuity of one or more mappings and containment of ranges amongst involved mappings.
involve mappings while the condition of \( t \)-norm \( T(t, t) = t \) for all \( t \in [0, 1] \) is replaced by an arbitrary continuous \( t \)-norm.

### 3.9 Corollary

Let \((X, M, T)\) be a fuzzy metric space and \(f, g\) be mappings from \(X\) into itself. Further, let the pair \((f, g)\) be weakly compatible and there exists a constant \(k \in \mathbb{R}_{>0}\) such that

\[
M(fx, fy, kt) \geq (4)
\]

holds for all \(x, y \in X\), \(a \in (0, 2), t > 0\) and \(\varphi \in \Phi\). If \((f, g)\) satisfies the (CLRg) property, then \(f\) and \(g\) have a unique common fixed point in \(X\).

**Proof:**

Take \(p = f\) and \(q = g\) in Theorem 3.3, then we get the result. □

### 3.10 Theorem

Let \((X, M, T)\) be a fuzzy metric space. Further, let the pair \((f, g)\) of self-mappings be weakly compatible satisfying inequality (4) of Corollary 3.9. If \(f\) and \(g\) satisfy E.A property and the range of \(g\) is a closed subspace of \(X\), then \(f\) and \(g\) have a unique common fixed point in \(X\).

**Proof:**

Since the pair \((f, g)\) satisfies E.A property, there exists a sequence \(\{x_n\}\) in \(X\) such that

\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = z
\]

for some \(z \in X\). It follows, from \(g(X)\) being a closed subspace of \(X\) that there exists \(u \in X\) such that \(z = gu\). Therefore \(f\) and \(g\) satisfy the (CLRg) property. From Corollary 3.9, the result follows. □

In what follows, we present some illustrative examples which demonstrate the validity of the hypothesis and degree of utility of our results.

### 3.11 Example

Let \(X = [2, 19)\) with the \(t\)-norm defined by \(d(x, y) = |x - y|\) and for each \(t \in [0, 1]\) define

\[
M(x, y, t) = \begin{cases} 
|x - y| & \text{if } t > 0 \\
0 & \text{if } t = 0
\end{cases}
\]

for all \(x, y \in X\). Clearly \((X, M, T)\) is a fuzzy metric space with \(t\)-norm defined by \(T(a, b) = \min\{a, b\}\) for all \(a, b \in [0, 1]\). Consider a function

\[
\varphi : [0, 1]^5 \to [0, 1]
\]

defined by

\[
\varphi(x_1, x_2, x_3, x_4, x_5) = \frac{1}{2} (\min\{x_i\})^2.
\]

Then we have \(M(fx, fy, t) \geq \varphi(x_1, x_2, x_3, x_4, x_5)\). Define the self-mappings \(f\) and \(g\) on \(X\) by

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2, if \( x \in \{2\} \cup (3, 19) \)

\[
\begin{align*}
fx &= 2, & \text{if } x = 2 \\
gx &= 12, & \text{if } x \in (2, 3]
\end{align*}
\]

\[
\begin{align*}
&\text{if } x \in (2, 3] \\
&\text{if } x \in (3, 19).
\end{align*}
\]

Taking \( \{x_n\} = 3 \) or \( \{x_n\} = \{2\} \), it is clear that the pair \((f, g)\)

satisfies the (CLRg) property since

\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = 2 = g(2) \in X.
\]

It is noted that \( f(X) = \{2, 15\} \subseteq \{2, 10\} \cup \{12\} = g(X) \). Thus, all the

conditions of Corollary 3.9 are satisfied for a fixed constant \( k \in 0, \frac{1}{2} \)

and 2 is a unique common fixed point of the self-mappings \( f \) and \( g \). Also, all the involved mappings are even discontinuous at their unique common fixed point 2. Here, it may be pointed out that \( g(X) \) is not a closed subspace of \( X \). □

3.12 Example

In the setting of Example 3.11, replace the mapping \( g \) by the following, besides retaining the rest:

\[
\begin{align*}
&\text{if } x = 2 \\
&\text{if } x \in (2, 3] \\
&\text{if } x \in (3, 19)
\end{align*}
\]

\[
\begin{align*}
gx &= 15, & \text{if } x \in (2, 3] \\
&\text{if } x \in (3, 19) \\
x + 1 \\
2
\end{align*}
\]
Taking \( \{ x \}_{n=1}^n = 3 + \frac{1}{n} \) or \( \{ x \}_{m=1}^m = \{ 2 \} \), it is clear that the pair \((f, g)\) satisfies the E.A property since

\[
\lim_{n \to \infty} f_{x_n} = \lim_{n \to \infty} g_{x_n} = 2 \in X.
\]

It is noted that \( f(X) = \{ 2, 15 \} \subseteq [2, 10] = g(X) \). Thus, all the conditions of Corollary 3.9 are satisfied and 2 is a unique common fixed point of the mappings \( f \) and \( g \). Notice that all the involved mappings are even discontinuous at their unique common fixed point 2. Here, it is worth noting that \( g(X) \) is a closed subspace of \( X \). \[\Box\]

Our next theorem extends Corollary 3.9 to two finite families of self-mappings which are pairwise commuting.

3.13 Theorem

Let \( \{ f_i \}_{i=1}^m \) and \( \{ g_j \}_{j=1}^n \) be two finite families of self-mappings in a fuzzy metric space \((X, M, T)\) such that \( f = f_1f_2...f_m \) and \( g = g_1g_2...g_n \) which satisfy the inequality (4) of Corollary 3.9. If the pair \((f, g)\) satisfies (CLRg) property, then \( f \) and \( g \) have a unique point of coincidence.

Moreover, \( \{ f_i \}_{i=1}^m \) and \( \{ g_j \}_{j=1}^n \) have a unique common fixed point

\( i \in \{1, 2, ..., m\} \) and \( j \in \{1, 2, ..., n\} \)

provided the pair of families \((\{ f_i \}, \{ g_j \})\) commute pairwise, where \( i \in \{1, 2, ..., m\} \) and \( j \in \{1, 2, ..., n\} \).

Proof:
The proof of this theorem can be completed on the lines of Theorem 3.1 contained in Imdad et al. [17], hence details are avoided. \[\Box\]

Putting \( f_1 = f_2 = ... = f_m = f \) and \( g_1 = g_2 = ... = g_n = g \) in

Theorem 3.13, we get followings result:

3.14 Corollary

Let \( f \) and \( g \) be two self-mappings of a fuzzy metric space \((X, M, T)\). Further, let the pair \((f^n, g^n)\) satisfies (CLRg) property.

Then there exists a constant \( k \in 0, \frac{1}{2} \) such that

\[
M (g^n x, g^n y, t), M (f^n x, f^n x, t), M (f^n y, g^n y, t),
\]

\[
M (f^n x, f^n y, k t) \geq \phi
\]

holds for all \( x, y \in X, a \in (0, 2), t > 0, \phi \in \Phi \) and \( m \) and \( n \) are fixed positive integers, then \( f \) and \( g \) have a unique common fixed point provided the pairs \((f^n, g^n)\) commute pairwise.

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3.15 Remark
Theorem 3.10, Theorem 3.13 and Corollary 3.14 can also be outlined in respect of Corollary 3.7.

3.16 Remark
Using Example 3.12, we can obtain several fixed point theorems in fuzzy metric spaces in respect of Theorems 3.10 and 3.13 and Corollaries 3.7, 3.9 and 3.14.

References