IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 2 Ver. V (Mar - Apr. 2015), PP 46-50
www.iosrjournals.org

General Class of Polynomials, I Function and $\overline{H}$-Function Associated With Feynman Integral

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Abstract: *In this paper we find certain new double integral relation pertaining to a product involving a general class of polynomials, I function and $\overline{H}$-function. These double integral relations are unified in nature and act as a key formulae from which we can obtain as their particular cases. The aim of present paper is to explain certain integral property of a general class of polynomial, $\overline{H}$ function and I function. Here we also discuss certain integral properties of a I function and $\overline{H}$-function, proposed by Inayat-Hussain which contain a certain class of Feynman integrals, the exact partition of a Gaussian model in Statistical Mechanics and several other functions as its particular cases.*

Keywords: *Feynman integrals, I function, $\overline{H}$-function, Hermite polynomials, Laguerre polynomials, general class of polynomial.*

I. Introduction

The $\overline{H}$-function [6] is a new generalization of the well known Fox's H-function [4]. The $\overline{H}$-function pertains the exact partition function of the Gaussian model in statistical mechanics, functions useful in testing hypothesis and several others as its particular cases. The conventional formulation may fail pertaining to the domain of quantum cosmology but Feynman path integrals apply [12,13]. Feynman integral are useful in the study and development of simple and multiple variable hypergeometric series which in turn are useful in statistical mechanics.

The I-function defined as

$$I[z]=I\left[\ z\left|\begin{matrix}(a_j,\alpha_j),(a_{ji},\alpha_{ji})\\(b_j,\beta_j),(b_{ji},\beta_{ji})\end{matrix}\right.\right]$$

$$=I^{m,n}_{p_i,q_i:\ell}\left[z\left|\begin{matrix}\{(a_j,\alpha_j)_{1,n}\},\{(a_{ji},\alpha_{ji})_{n+1,p_i}\\\{(b_j,\beta_j)_{1,m}\},\{(b_{ji},\beta_{ji})_{m+1,q_i}\}\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi}\oint\theta(s)z^s\,ds$$

where

$$\theta(s)=\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_js)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_js)}{\sum_{i=1}^{\ell}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s)\right\}} \qquad \ldots(1.1)$$

The $\overline{H}$-function will be defined and represent as given in [1]

$$\overline{H}_{P,Q}^{M,N}[x]=\overline{H}_{P,Q}^{M,N}\left[x\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j)_{M+1,Q}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\bar{\phi}(\xi)\,x^{\xi}\,d\xi \qquad \ldots(1.2)$$

where

$$\bar{\phi}(\xi)=\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}A_j}{\prod_{j=M+1}^{Q}\{(\Gamma 1-b_j+\beta_j\xi)\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \qquad \ldots(1.3)$$

Which contains fractional powers of some of the gamma functions. Here a_j (j = 1,…,P) and b_j (j = 1,…,Q) are complex parameters, $a_j \geq 0$ (j = 1,…,P), $\beta_j \geq 0$ (j = 1,…,Q) (not all zero simultaneously and the exponents A_j (j = 1,…,N) and B_j (j = M+1,…,Q) can take on non-integer values. The contour in (1.2) is imaginary axis $R(\xi) = 0$. It is suitably indented in order to avoid the singularities of the gamma functions and to keep those singularities on appropriate side. Again for A_j (j=1,…,N) not an integer, the poles of the gamma function of the numerator in (1.3) are converted to branch points. However, a long as there is no coincidence of pole from any $\Gamma(b_j-\beta_j\xi)(j=1,\ldots,M)$ and $\Gamma(1-a_j+\alpha_j\xi)(j=1,\ldots,N)$ pair, the branch cuts can be chosen so that the path of integration can be distorted in the useful manner. For the sake of brevity

$$T=\sum_{j=1}^{M}\beta_j+\sum_{j=1}^{N}A_j\alpha_j-\sum_{j=M+1}^{Q}B_j\beta_j-\sum_{j=N+1}^{P}\alpha_j>0$$

The general class of polynomial introduced by Srivastava [12]

$$S_n^m[x]=\sum_{s=0}^{[n/m]}\frac{(-n)_{ms}}{s!}A_{n,s}X^s,\; n=0,1,2 \qquad \ldots(1.4)$$

II. Main Result

(A) We will obtain the following result:

$$\int_0^1\int_0^1\left(\frac{1-p}{1-pq}q\right)^l\left(\frac{1-q}{1-pq}\right)^m\frac{1-pq}{(1-p)(1-q)}I\left[\frac{1-p}{1-pq}wq\right]S_n^m\left[\frac{1-p}{1-pq}wq\right]\overline{H}_{P,Q}^{M,N}\left[\frac{1-qw}{1-pq}\right]dp\,dq$$

$$=\sum_{K=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\,w^k\,\Gamma(l+s+k)\,I\left[w\left|\begin{matrix}(a_j,\alpha_j),(a_{ji},\alpha_{ji})\\(b_j,\beta_j),(b_{ji},\beta_{ji})\end{matrix}\right.\right]$$

$$\overline{H}_{P+1,Q+1}^{M,N+1}\left[\begin{matrix}(1-m::1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j:B_j)_{M+1,Q},(1-l-m-k-s:1)\end{matrix}\right|w\Bigg] \qquad \ldots(2.1)$$

provided that $R[\alpha+\beta+b_j/\beta_j]>0, |\arg V|<\frac{1}{2}T\pi,$

Proof. We have

$$S_n^m\left[\frac{1-p}{1-pq}wq\right]I\left[\frac{1-p}{1-pq}wq\right]\bar{H}_{P,Q}^{M,N}\left[\frac{1-q}{1-pq}w\right]$$

$$=\sum_{K=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\left(\frac{1-p}{1-pq}wq\right)^k\frac{1}{2\pi}\oint\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma 1-a_j+\alpha_j s}{\sum_{i=1}^{\ell}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s)\right\}}$$

$$.\left(\frac{1-p}{1-pq}wq\right)^s ds\,\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}A_j}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j\xi)\}\beta_j\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)}\left[\frac{1-q}{1-pq}w\right]^{\xi}d\xi$$

…(2.2)

Multiplying both sides of (2.2) by $\left[\frac{1-p}{1-pq}q\right]^l\left[\frac{1-q}{1-pq}\right]^m\left[\frac{1-pq}{(1-p)(1-q)}\right]$

and integration with respect to p and q between 0 and 1 for both the variable and making a use of a known result [2, p.145],we get the required result (2.1) after a little simplification

(B) $$\int_0^{\infty}\int_0^{\infty}f(w+z)\,w^{l-1}z^{m-1}S_n^m(w)I(w)\bar{H}_{P,Q}^{M,N}[z]\,dw\,dz$$

$$=\Gamma(l+k+s)\sum_{K=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}I\left[u\left|\begin{matrix}(a_j,\alpha_j),(a_{ji},\alpha_{ji})\\(b_j,\beta_j),(b_{ji},\beta_{ji})\end{matrix}\right.\right]\int_0^{\infty}f(u)\,u^{l+m+k-1}$$

$$.\bar{H}_{P+1,Q+1}^{M,N+1}\left[\begin{matrix}(1-m:1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j:B_j)_{M+1,Q},(1-l-m-k-s:1)\end{matrix}\left|u\right.\right]du\quad\text{…(2.3)}$$

provided that $R(\alpha+\beta+b_j/\beta_j)>0.$

Proof. Using (1.1), (1.2) and (1.3), we have

$$S_n^m(w)I(w)\bar{H}_{P,Q}^{M,N}[z]$$

$$=\sum_{K=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}w^k\frac{1}{2\pi}\int\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma 1-a_j+\alpha_j s}{\sum_{i=1}^{\ell}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_j+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s)\right\}}w^s\,ds$$

$$. \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j\xi)\prod_{j=1}^{N} \{\Gamma(1-a_j+\alpha_j\xi)\}A_j}{\prod_{j=M+1}^{Q} \{\Gamma(1-b_j+\beta_j\xi)\}\beta_j \prod_{j=N+1}^{P} \Gamma(a_j-\alpha_j\xi)} z^{\xi}\, d\xi \quad \ldots(2.4)$$

Multiplying both side by $f(w+z)\, w^{l-1}\, z^{m-1}$ and integrating with respect to w and z between 0 and ∞ for both the variable and make a use of a known result [2, p.177], we get the required result. Letting $f(u)=e^{-pu}$ in (2.3), we get the particular case after simplification

(c) $$\int_0^1 \int_0^1 \phi(wz)(1-w)^{l-1}(1-z)^{m-1} z^l \; I[(z(1-w)]\, S_n^m[(z(1-w)]\overline{H}_{P,Q}^{M,N}[1-z]\,dw\,dz$$

$$=\Gamma(l+k+s)\sum_{K=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\; I\left[(1-u)\left|\begin{matrix}(a_j,\alpha_j),(a_{ji},\alpha_{ji})\\(b_j,\beta_j),(b_{ji},\beta_{ji})\end{matrix}\right.\right]\int_0^1 f(u)(1-u)^{\alpha+k+s+\beta-1}$$

$$.\ \overline{H}_{P+1,Q+1}^{M,N+1}\left[(1-u)\left|\begin{matrix}(1-m:1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j:B_j)_{M+1,Q},(1-s-k-l-m:1)\end{matrix}\right.\right]du \quad \ldots(2.5)$$

provided that R(α) > 0, R(β) > 0.

Proof. Using equation (1.1) and (1.2), (1.3) we have

$$S_n^m[(z(1-w)]I[z(1-w)]\,\overline{H}_{P,Q}^{M,N}[1-z]=\sum_{K=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}[z(1-w)]^k$$

$$\frac{1}{2\pi}\int \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma 1-a_j+\alpha_j s}{\sum_{i=1}^{\ell}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_j+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s)\right\}} z^s(1-w)^s\, ds$$

$$. \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^{M} \Gamma(b_j - \beta_j\xi)\prod_{j=1}^{N} \{\Gamma(1-a_j+\alpha_j\xi)\}A_j}{\prod_{j=M+1}^{Q} \{\Gamma(1-b_j+\beta_j\xi)\}\beta_j \prod_{j=N+1}^{P} \Gamma(a_j-\alpha_j\xi)} (1-z)^{\xi}\, d\xi \quad \ldots(2.6)$$

Multiplying both side of (2.6) by $\phi(wz)(1-w)^{l-1}(1-z)^{m-1}z^l$ and integrating with respect to w and z between 0 and 1 for both the variable and use of result [2, p.243] and by further simplification, we get the result (2.5).

Letting f(u) = u^{m-1} in (2.5), we get the particular result after simplification.

III. Special cases

(i) By applying our results given in (2.1), (2.3) (2.5) to the case of hermite polynomial [13] and [14] and by setting

$$S_n^2[x] \to x^{\frac{n}{2}} H_n\left[\frac{1}{2\sqrt{x}}\right]$$

In which case $m = 2$ $A_{n,k} = (-1)^k$, we have the following exciting consequences of the main results.

(A.1)

$$\int_0^1\int_0^1 \left(\frac{1-p}{1-pq}q\right)^l \left(\frac{1-q}{1-pq}\right)^m \left[\frac{1-pq}{(1-p)(1-q)}\right] I\left[\frac{1-p}{1-pq}wq\right]\left[\frac{1-p}{1-pq}wq\right]^{n/2}$$

$$H_n\left[\frac{1}{2\sqrt{\frac{1-p}{1-pq}wq}}\right]\overline{H}_{P,Q}^{M,N}\left[\frac{1-qw}{1-pq}\right]dp\,dq$$

$$=\sum_{K=0}^{[n/2]}\frac{(-n)_{2k}}{k!}(-1)^k w^k\Gamma(l+k+s)I\left[w\left|\begin{matrix}(a_j,\alpha_j),(a_{ji},\alpha_{ji})\\(b_j,\beta_j),(b_{ji},\beta_{ji})\end{matrix}\right.\right]$$

$$\overline{H}_{P+1,Q+1}^{M,N+1}\left[\begin{matrix}(1-m::1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j:B_j)_{M+1,Q},(1-l-m-k-s:1)\end{matrix}\middle|w\right]$$

Valid under the same conditions as essential for (2.1)

(A.2) $$\int_0^\infty\int_0^\infty f(w+z)\,w^{l+n/2}\,H_n\left[\frac{1}{2\sqrt{w}}\right]\overline{H}_{P,Q}^{M,N}[z]\,dw\,dz$$

$$=\Gamma(l+k+s)\sum_{K=0}^{[n/2]}\frac{(-n)_{2k}}{k!}(-1)^k I\left[u\left|\begin{matrix}(a_j,\alpha_j),(a_{ji},\alpha_{ji})\\(b_j,\beta_j),(b_{ji},\beta_{ji})\end{matrix}\right.\right]\int_0^\infty f(u)\,u^{l+m+k-1}$$

$$\overline{H}_{P+1,Q+1}^{M,N+1}\left[\begin{matrix}(1-m:1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j:B_j)_{M+1,Q},(1-l-m-k-s:1)\end{matrix}\middle|u\right]du$$

Valid under the same conditions as essential for (2.1)

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