Fixed Point Result Satisfying $\Phi$ - Maps in G-Metric Spaces

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Abstract: In this paper, we elaborate some existing result of fixed point theorem, that fulfill the nature of G-metric space and satisfy the $\Phi$-maps. Previously Erdal Karapinar and Ravi Agrawal [24] have modified some existing result of fixed point theory of Samet et al Int. J. Anal. (2013:917158, 2013) [44] and Jleli-Samet (Fixed point theory application. 2012:2010,2012) [45] in a different way.

I. Introduction

The concept of G-metric spaces was introduced by Mustafa and Sims [25], G-metric spaces is generalization of a metric spaces (X,d). In this paper they characterized the Banach contraction mapping principal [10] in the context of G-metric spaces. Subsequently many fixed point result on such spaces appeared. Since one is adapted from other. The G-metric spaces is to understand the geometry of three points instead of two. Many result are obtained by contraction condition.

In 2013, Samet et al [38] and Jleli Samet [39] observed that some fixed point theorems in the context of a G-metric space in literature can be concluded by some existing results in the setting of (quasi)metric spaces. Also the contraction condition of the fixed point theorem on a G-metric space can be reduced to two variables instead of three. In [20,38,39] the authors find d(x,y) = G(x,y) form a quasi-metric. Erdal Karapinar and Ravi Agrawal modified some existing result to suggest new fixed point theorem, in this way they approach (Samet et al and Jleli Samet) in a different technique.

2. Definition 2.1 (See [1]) Let X be a non-empty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties:

(G1) $G(x,y,z) = 0$ if $x = y = z$,
(G2) $0 < G(x,y,z)$ for all $x, y \in X$ with $x \neq y$,
(G3) $G(x,y,z) \leq G(x,y,z)$ for all $x, y, z \in X$ with $x \neq y$,
(G4) $G(x,y,z) = G(y,z,x) = \cdots$ (symmetry in all three variables),
(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function $G$ is called a generalized metric or, more specifically, a G-metric on X, and the pair $(X,G)$ is called a G-metric space.

Every G-metric on X defines a metric $d_G$ on X by

$d_G(x,y) = G(x,y) + G(y,x)$ for all $x, y \in X$.

Example 1 Let $(X,d)$ be a metric space. The function $G : X \times X \to [0, +\infty)$, defined as

$G(x,y,z) = \max \{d(x,y),d(y,z),d(z,x)\}$

Or

$G(x,y,z) = d(x,y) + d(y,z) + d(z,x)$, for all $x, y, z \in X$, is a G-metric on X.

Definition 2.2 Let $(X,G)$ be a G-metric space, and let $(x_n)$ be a sequence of points of $X$. We say that $(x_n)$ is G-convergent to $x \in X$ if

$\lim_{n,m \to +\infty} G(x_n, x_m) = 0$.

That is, for any $\epsilon > 0$, there exists $N \in N$ such that $G(x_n, x_m) < \epsilon$ for all $n, m \geq N$. We call $x$ the limit of the sequence and write $x_n \to x$ or $\lim n \to +\infty x_n = x$.

Proposition 2.1 Let $(X,G)$ be a G-metric space. The following are equivalent:

(1) $(x_n)$ is G-convergent to x,
(2) $G(x_n,x) \to 0$ as $n \to +\infty$,
(3) $G(x_n,x) \to 0$ as $n \to +\infty$,
(4) $G(x_n,x) \to 0$ as $n,m \to +\infty$.

Definition 2.3 Let $(X,G)$ be a G-metric space. A sequence $(x_n)$ is called a G-Cauchy sequence.
if, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, xm, x_l) < \varepsilon$ for all $m, n, l \geq N$, that is, $G(x_n, xm, x_l) \rightarrow 0$ as $n, m, l \rightarrow +\infty$.

**Proposition 2.2** Let $(X, G)$ be a $G$-metric space. Then the following are equivalent:
1. the sequence $\{x_n\}$ is $G$-Cauchy,
2. for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, xm, x_l) < \varepsilon$ for all $m, n \geq N$.

**Definition 2.4** A $G$-metric space $(X, G)$ is called $G$-complete if every $G$-Cauchy sequence is $G$-convergent in $(X, G)$.

**Lemma 2.1** Let $(X, G)$ be a $G$-metric space. Then $G(x, y, z) \leq 2G(x, y, y)$ for all $x, y \in X$.

**Definition 2.5** Let $(X, G)$ be a $G$-metric space. A mapping $T : X \rightarrow X$ is said to be $G$-continuous if $\{T(x_n)\}$ is $G$-convergent to $T(x)$ where $\{x_n\}$ is any $G$-convergent sequence converging to $x$.

In [22], Mustafa characterized the well-known Banach contraction mapping principle in the context of $G$-metric spaces in the following ways.

**Theorem 2.1** Let $(X, G)$ be a complete $G$-metric space and let $T : X \rightarrow X$ be a mapping satisfying the following condition for all $x, y, z \in X$:

$G(Tx, Ty, Tz) \leq k G(x, y, z),
$ Where $k \in [0, 1)$. Then $T$ has a unique fixed point.

**Theorem 2.2** Let $(X, G)$ be a complete $G$-metric space and let $T : X \rightarrow X$ be a mapping satisfying the following condition for all $x, y \in X$:

$G(Tx, Ty, Ty) \leq k G(x, y, y),
$ where $k \in [0, 1)$. Then $T$ has a unique fixed point.

**Theorem 2.3** Let $(X, G)$ be a $G$-metric space. Let $T : X \rightarrow X$ be a mapping such that $G(Tx, Ty, Tz) \leq a G(x, y, z) + b G(x, Tx, Tx) + c G(y, Ty, Ty) + d G(z, Tz, Tz)$ for all $x, y, z, \in X$ where $a, b, c, d$ are positive constants such that $k = a + b + c + d < 1$. Then there is a unique $x \in X$ such that $Tx = x$.

**Theorem 2.4** Let $(X, G)$ be a $G$-metric space. Let $T : X \rightarrow X$ be a mapping such that $G(Tx, Ty, Tz) \leq k \{ G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz) \}$ for all $x, y, z, \in X$ where $k \in [0, \frac{3}{4})$. Then there is a unique $x \in X$ such that $Tx = x$.

**Theorem 2.5** Let $(X, G)$ be a $G$-metric space. Let $T : X \rightarrow X$ be a mapping such that $G(Tx, Ty, Tz) \leq a G(x, y, z) + b G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)$ for all $x, y, z, \in X$ where $a, b$ are positive constants such that $k = a + b < 1$. Then there is a unique $x \in X$ such that $Tx = x$.

**Theorem 2.6** Let $(X, G)$ be a $G$-metric space. Let $T : X \rightarrow X$ be a mapping such that $G(Tx, Ty, Tz) \leq a G(x, y, z) + b \max \{ G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz) \}$ for all $x, y, z, \in X$ where $a, b$ are positive constants such that $k = a + b < 1$. Then there is a unique $x \in X$ such that $Tx = x$.

**Theorem 2.7** Let $(X, G)$ be a $G$-metric space. Let $T : X \rightarrow X$ be a mapping such that $G(Tx, Ty, Tz) \leq k \max \{ G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz), G(z, Tx, Tz), G(x, Ty, Ty), G(y, Tz, Tz) \}$ for all $x, y, z, \in X$ where $k \in [0, \frac{1}{2})$. Then there is a unique $x \in X$ such that $Tx = x$.

**Theorem 2.8** Let $(X, G)$ be a complete $G$-metric space and let $T : X \rightarrow X$ be a given mapping satisfying $G(Tx, Ty, Tz) \leq G(x, y, z) - \phi(G(x, y, z))$ for all $x, y \in X$ where $\phi : [0, +\infty) \rightarrow [0, +\infty)$ is continuous with $\phi - 1(\{0\}) = 0$. Then there is a unique $x \in X$ such that $Tx = x$.
**Definition 2.6** A quasi-metric on a nonempty set $X$ is a mapping $p : X \times X \rightarrow [0, \infty)$ such that

(p1) $x = y$ if and only if $p(x, y) = 0$,
(p2) $p(x, y) \leq p(x, z) + p(z, y)$,

for all $x, y, z \in X$. A pair $(X, p)$ is said to be a quasi-metric space.

Samet et al. and Jleli-Samet noticed that $p(x, y) = pG(x, y) = G(x, y, y)$ is a quasimetric whenever $G : X \times X \times X \rightarrow [0, \infty)$ is a G-metric. It is well known that each quasimetric induces a metric. Indeed, if $(X, p)$ is a quasi-metric space, then the function defined by $d(x, y) = dG(x, y) = \max \{p(x, y), p(y, x)\}$ for all $x, y \in X$ is a metric on $X$.

**Theorem 2.9** Let $(X, d)$ be a complete metric space and let $T : X \rightarrow X$ be a mapping with the property $d(Tx, Ty) \leq q \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$ for all $x \in X$, where $q$ is a constant such that $q \in [0, 1)$. Then $T$ has a unique fixed point.

**Proposition 2.3**

(A) If $(X, G)$ is a complete G-metric space, then $(X, d)$ is a complete metric space.

(B) If $(X, G)$ is a sequentially G-compact G-metric space, then $(X, d)$ is a compact metric space.

**II. Main Result**

**Theorem 3.1** Let $(X, G)$ be a complete G-metric space and let $f : X \rightarrow X$ be a given mapping satisfy for all $x, y \in X$, where $\phi : [0, \infty) \rightarrow [0, \infty)$ is continuous with $\phi(0) = 0$, then there is a unique $x \in X$ st. $f(x) = x$.

**Proof:** We first show that if the fixed point of the operator $f$ exist, then it is unique. Suppose, on contrary, that $x$ and $y$ are two fixed point of $f$, such that $x \neq y$, hence $G(x, x, y) \neq 0$.

From equation (1), we get

$$G(fx, f^2y, f^2z) \leq G(x, fy, fz) - \phi(G(x, fy, fz))$$

Which is equivalent to

$$G(x, y, y) \leq G(x, y, y) - \phi(G(x, y, y))$$

A contradiction hence $f$ has a unique fixed point.

Let $x_n \in X$, we define a sequence $\{x_n\}$ by $x_{n+1} = f(x_n)$, for some $n \in \mathbb{N}$.

If $x_n = x_{n+1}$, for some $n$, then $f$ has a fixed point.

Taking $x_n = x_{n+1}$, $y = z = x_n$.

Now from equation (1), we have

$$G(x_n, x_{n+1}, x_{n+1}) = G(fx_{n-1}, f^2x_{n-1}, f^2x_{n-1})$$

$$\leq G(x_{n-1}, fx_{n-1}, fx_{n-1}) - \phi(G(x_{n-1}, fx_{n-1}, fx_{n-1}))$$

This shows that $\{G(x_n, x_{n+1}, x_{n+1})\}$ converges to $s \geq 0$.

We shall show that $s = 0$.

Suppose, on contrary that $s > 0$.

Letting $n \rightarrow \infty$ in equation (2), We get $s = 0$.

It is a contradiction. Hence conclude that $\lim_{n \rightarrow \infty} G(x_n, x_{n+1}, x_{n+1}) = 0$.

By lemma [2.1], we know that $\lim_{n \rightarrow \infty} G(x_n, x_{n+1}, x_{n+1}) = 0$.

Hence

$$\lim_{n \rightarrow \infty} G(x_n, x_{n+1}, x_{n+1}) = 0, n \rightarrow \infty$$

**References**

 DOI: 10.9790/5728-11243339  www.iosrjournals.org 35 | Page
Now next we show that the \( \{x_n\} \) is G-cauchy, on contrary let \( \{x_n\} \) is not G-cauchy sequence then so there exist \( \epsilon > 0 \) and subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) with \( n(k) > m(k) > k \).

Such that \( G(x_{n_k}, x_{m_k}, x_{m_k}) \geq \epsilon \) for all \( k \in \mathbb{N} \) ........................................ (4)

More over, corresponding to \( m_k \), we can choose \( n_k \), such that it is the smallest integer with \( n_k > m_k \) satisfying equation (4).

Then that \( G(x_{n_k}, x_{m_k}, x_{m_k}) \leq \epsilon \) ........................................ (5)

Then we have ,

\[
\epsilon \leq G(x_{n_k}, x_{m_k}, x_{m_k}) \\
\leq G(x_{n_k}, x_{m_k}, x_{m_k}) + G(x_{m_k}, x_{m_k}, x_{m_k}) \\
= \epsilon + G(x_{n_k}, x_{m_k}, x_{m_k})
\]

Setting \( k \rightarrow \infty \) and using equation (3), \( \lim k \rightarrow \infty G(x_{n_k}, x_{m_k}, x_{m_k}) = \epsilon \)

Now \( G(x_{n_k}, x_{m_k}, x_{m_k}) \leq G(x_{n_k}, x_{m_k}, x_{m_k}) + G(x_{n_k}, x_{m_k}, x_{m_k}) + G(x_{m_k}, x_{m_k}, x_{m_k}) \)

And

\( G(x_{n_k}, x_{m_k}, x_{m_k}) \leq G(x_{n_k}, x_{m_k}, x_{m_k}) + G(x_{m_k}, x_{m_k}, x_{m_k}) + G(x_{m_k}, x_{m_k}, x_{m_k}) \)

Setting \( k \rightarrow \infty \) in above inequality and using (3) and (5)

\( \lim k \rightarrow \infty G(x_{n_k}, x_{m_k}, x_{m_k}) = \epsilon \)

Now again from equation (1) and (4), we have

\[
\epsilon \leq \epsilon + G(x_{n_k}, x_{m_k}, x_{m_k}) + G(x_{n_k}, x_{m_k}, x_{m_k}) + G(x_{m_k}, x_{m_k}, x_{m_k})
\]

Letting \( k \rightarrow \infty \), we have \( \epsilon \leq \epsilon - G(\epsilon) \), Which is a contradiction, if \( \epsilon > 0 \).

So , we must have \( \epsilon = 0 \). This shows that \( \{x_n\} \) is G-cauchy sequence in \( X \). Since \( X \) is complete G-metric space .

So there exists \( z \in X \), such that \( \lim n \rightarrow \infty x_n \rightarrow z \).

Now we claim that \( f(z) = z \).

Consider \( G(z, f(z), f(z)) = G(f(z), f(z), f(z)) \)

\[
\leq G(z, f(z), f(z)) - G(f(z), f(z), f(z)) \\
= G(z, f(z), f(z)) - G(f(z), f(z), f(z))
\]

Let \( n \rightarrow \infty \), we get \( G(f(z), z, z) \leq 0 \) \( G(z, z, z) \)

\( = 0 \)

Hence \( G(f(z), z, z) = 0 \), i.e. \( f(z) = z \).

Hence \( z \) is a fixed point.

**Theorem 3.2:** Let \((X, G)\) be a G-metric space . Let \( f: X \rightarrow X \) be a mapping such that

\[
G(f(x), f(y), f(z)) \leq kM(x, y, z) \text{ for all } x, y, z \in X \text{ and } k \in [0.1) \text{ and}
\]

\[
M(x, y, z) = \max \{ G(x, y, z), G(f(x), f(y), f(z)), G(z, f(x), f(z)), G(z, f(y), f(z)), G(x, z, f(z)), G(y, f(z), f(z)), G(z, f(z), f(z)), G(f(z), f(z), f(z)) \}
\]

Then there is a unique \( x \in X \) such that \( f(x) = x \).

**Proof:** Let \( x_0 \in X \). We define \( \{x_n\} \) in the following \( f(x_n) = x_{n+1} \), \( n \in \mathbb{N} \)

Taking \( x = x_n, y = z = x_{n+1} \), we get

\( G(f(x_n), f(x_{n+1}), f(x_{n+1})) \leq kM(x_n, x_{n+1}, x_{n+1}) \)

Where

\[
M(x_n, x_{n+1}, x_{n+1}) = \max \{ G(x_n, x_{n+1}, x_{n+1}), G(f(x_n), f(x_{n+1}), f(x_{n+1})), G(x_{n+1}, f(x_n), f(x_{n+1})) \}
\]

DOI: 10.9790/5728-11243339  www.iosrjournals.org 36 | Page
Fixed Point result Satisfying $\phi$ - Maps in G-metric Spaces

$G(f x_{n+1}, f^2 x_n, f x_{n+1})$
$G(x_n, f x_n, f x_n), G(x_{n+1}, f x_{n+1}, f x_{n+1})$
$G(x_n, f x_n, f x_n), G(x_{n+1}, f x_{n+1}, f x_{n+1})$
$= \max \{ G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}) \}$
$= \max \{ G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}) \}$
$= \max \{ G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}) \}$

Case (i)- First let $M(x_n, x_{n+1}, x_{n+1}) = G(x_{n+1}, x_{n+1}, x_{n+1})$

By $G_5$, we get from above
$G(x_{n+1}, x_{n+1}, x_{n+1}) = G(f x_n, f x_n, f x_n)$
$\leq kM(x_n, x_{n+1}, x_{n+1})$
$= kG(x_{n+1}, x_{n+1}, x_{n+1})$
$\leq k^2G(x_{n+1}, x_{n+1}, x_{n+1})$

Which is a contradiction, since $0 \leq k < 1$.

Case (ii)- If $M(x_n, x_{n+1}, x_{n+1}) = G(x_{n+1}, x_{n+1}, x_{n+1})$

Then we get $G(x_{n+1}, x_{n+1}, x_{n+1}) = G(f x_n, f x_n, f x_n)$
$\leq kM(x_n, x_{n+1}, x_{n+1})$
$= kG(x_{n+1}, x_{n+1}, x_{n+1})$
$\leq k^2G(x_{n+1}, x_{n+1}, x_{n+1})$

This is a contradiction, since $0 \leq k < 1$.

Case (iii)- If $M(x_n, x_{n+1}, x_{n+1}) = G(x_{n+1}, x_{n+1}, x_{n+1})$

Then we get, $G(x_{n+1}, x_{n+1}, x_{n+1}) \leq kG(x_{n+1}, x_{n+1}, x_{n+1})$

Continuing in this way, we get
$G(x_{n+1}, x_{n+1}, x_{n+1}) \leq k^nG(x_1, x_1, x_1)$

Again,
$G(x_{n+1}, x_{n+1}, x_{n+1}) \leq k^nG(x_1, x_1, x_1)$

Let $n, m \to \infty$ we get, $G(x_n, x_m, x_m) \to 0$.

Hence $\{ x_n \}$ is a Cauchy sequence in $X$. Since $(X, G)$ is $G$-complete, then there exist $z \in X$ s.t. $\{ x_n \}$ is $G$-converges to $z$. Let on contrary that $z \neq f z$ for this let $x_{n+1} = f x_n$

$G(x_{n+1}, f z, f z) = G(f x_n, f z, f z)
\leq kM(x_n, z, z)$

Where
$M(x_n, z, z) = \max \{ G(x_n, z, z), G(f x_n, f z, f z), G(z, f x_n, f z), G(z, z, f x_n), G(z, f^2 x_n, f x_n), (z, f z, f z), G(x_n, f z, f z), G(x_n, f z, f z), G(x_n, f z, f z), G(x_n, f ^2 z, f z), (z, f ^2 z, f z), (f x_n, f ^2 z, f z) \}$

$= \max \{ G(x_n, z, z), G(f x_n, f z, f z), G(z, f x_n, f z), G(z, z, f x_n), G(z, x_n, f z), G(x_n, x_n, f z), G(x_n, x_n, f z), G(x_n, x_n, f z), (z, x_n, f z), (x_n, x_n, f z), (z, x_n, f z), (x_n, x_n, f z) \}$

Letting $n \to \infty$ since $G$ is continuous, we get
$G(z, f z, f z) \leq kG(z, f z, f z)$

Or
$G(z, f z, f z) \leq kG(z, f z, f z)$
$\leq k[G(z, f z, f z) + G(f z, z, f z)]$
$= k^2G(z, f z, f z)$

This is a contradiction.

Since $0 \leq k < 1$. So $f z = z$.

Uniqueness:- Next we show that uniqueness of $z$ of $f$. Suppose on contrary, there exist another common fixed point $u \in X$ with $z \neq u$.

We get $G(z, z, u) = G(f z, f z, f u)$
$\leq kM(z, z, u)$

DOI: 10.9790/5728-11243339 www.iosrjournals.org 37 | Page
Fixed Point result Satisfying φ - Maps in G-metric Spaces

We get a contradiction, since $0 \leq k < 1$. Thus $z = u$ is a fixed point.

Example:- Let $X = [0, \infty)$, $G: X \times X \times X \to \mathbb{R}$ be defined by

$$G(x, y, z) = \begin{cases} 0, & \text{if } x = y = z \\ \max \{ x, y, z \}, & \text{otherwise} \end{cases}$$

Then $(X, G)$ is a complete G-metric space.

Let $f: X \to X$ be defined by

$$f(x) = \begin{cases} \frac{1}{2}x, & \text{if } 0 \leq x < \frac{1}{2} \\ \frac{1}{3}x^3, & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

And $\mathcal{O}(x) = \frac{1}{2}t$, for all $t \in [0, \infty)$

Solution:- First we examine the following cases:

Let $0 \leq x, y < \frac{1}{2}$, then

$$G(fx, fy, fz) = \max \left\{ \frac{1}{3}x, \frac{1}{3}y, \frac{1}{3}z \right\} \leq \frac{1}{3} \max \left\{ x, y, z \right\}$$

Let $\frac{1}{2} \leq x, y < 1$, then

$$G(fx, fy, fz) = \max \left\{ \frac{1}{3}x^3, \frac{1}{3}y^3, \frac{1}{3}z^3 \right\} \leq \frac{1}{3} \max \left\{ x^3, y^3, z^3 \right\}$$

Let $0 \leq x < \frac{1}{2} \leq y < 1$, then

$$G(fx, fy, fz) = \max \left\{ \frac{1}{3}x, \frac{1}{3}y^2, \frac{1}{3}z \right\} \leq \frac{1}{3} \max \left\{ x, y^2, z \right\}$$

Let $0 \leq y < \frac{1}{2} \leq x < 1$, then

$$G(fx, fy, fz) = \max \left\{ \frac{1}{3}y^3, \frac{1}{3}x^3, \frac{1}{3}z \right\} \leq \frac{1}{3} \max \left\{ y^3, x^3, z \right\}$$

Hence $f$ has a unique fixed point.

Here $(0,0,0)$ is a fixed point.

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Fixed Point result Satisfying $\phi$ - Maps in G-metric Spaces


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