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Numerical Solution of Two Dimensional Diffusion Equations with Nonlocal Boundary Conditions by Iterative Laplace Transform Method

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Abract: *In recent papers solution Two â€“Dimensional Diffusion Equations of partial differential equations with nonlocal boundary conditions was introduced using iterative laplace transform method. This method which combines two method iterative method and laplace transform method is successfully implemented these numerical schemes for both Homogeneous and Inhomogeneous cases of the important equation comparsion with Special Class of Pade Approximants ,the numerical results of ILTM show that based numerical schemes are quite accurate and easily implemented*
Keywords: *Partial Differential Equations, Laplace Transform Method, Iterative Method.*

I. Introduction

Partial differential equations arise in formulations of problems involving functions of several variables such as the propagation of sound or heat, electrostatics, electrodynamics, fluid flow, and elasticity, [2].Parabolic partial differential equations with nonlocal boundary conditions arise in modeling of a wide range of important application areas such as chemical diffusion, thermoelasticity,heat conduction process, control theory and medicine science[3], in this paper introduce non classical initial boundary value problems that is the solution of Two â€“Dimensional Diffusion Equations of partial differential equations with nonlocal boundary conditions. In 2006, Daftardar-Gejji and Jafari proposed a new iterative method to seek numerical solutions of nonlinear functional equations[4],[5]Jafari et al. firstly applied Laplace transform in the iterative method and proposed anew direct method called iterative Laplace transform method to search for numerical solutions partial differential equations [6].The method is based on Laplace transform, iterative method. Jafari and Seifi successfully obtained the numerical solutions of Jafari and Seifi successfully obtained the numerical solutions of two systems of space-time fractional differential equations [8].

II. Basic Idea of Iterative Laplace Transform Method[8]

In this section, the aim idea of the iterative Laplace transform method [6] consider the general two â€“dimensional diffusion equations of partial differential equation of the form.

$$\frac{\partial u}{\partial t}=\alpha F(u,\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2});\ 0<x,y<1,t>0 \qquad (1)$$

with initial value condition

$$u^{(k)}(x,y,0)=g_k(x,y) \qquad (2)$$

where g is continuous function , $u=u(x,y,t)$ and $F(u,\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2})$ is linear or nonlinear operator of $u,\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2}$ .Taking Laplace transfer of both sides of (Eq.(1))results in:

$$s\mathrm{L}\{u(x,y,t)\}-\sum_{k=0}^{m-1}s^{-k}u^{(k)}(x,y,0)=\mathrm{L}\{\alpha\ F(u,\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2})\} \qquad (3)$$

given by:

$$\mathrm{L}\{u(x,t)\}=\sum_{k=0}^{m-1}s^{-1-k}u^{(k)}(x,y,0)+s^{-1}\mathrm{L}\{\alpha\ F(u,\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2})\} \qquad (4)$$

Operating with Laplace inverse (denoted by L^{-1} throughout the present paper)0n the both sides of (Eq.(4))gives:

$$u(x,y,t)=L^{-1}\{\sum_{k=0}^{m-1}s^{-1-k}u^{(k)}(x,y,0)\}+L^{-1}\{s^{-\alpha}L\{\alpha\, F(u,\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2})\}\} \qquad (5)$$

Which can be written (Eq.(5))given by form:

$$u(x,y,t)=L^{-1}\{\sum_{k=0}^{m-1}s^{-1-k}u^{(k)}(x,y,0)\}+N(u,\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2}) \qquad (6)$$

Where $N(u,\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2})=L^{-1}\{s^{-\alpha}L\{\alpha\, F(u,\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2})\}\}$.

The iterative Laplace transform method represents the solution as an infinite series:

$$u(x,y,t)=\sum_{n=0}^{\infty}u_n \qquad (7)$$

where the terms u_n are to be recursively computed. The linear or nonlinear operator $N(u,\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2})$ can be computed as follows:

$$\begin{aligned} N(\sum_{n=0}^{\infty}u_n,\sum_{n=0}^{\infty}\frac{\partial^2 u_n}{\partial x^2},\sum_{n=0}^{\infty}\frac{\partial^2 u_n}{\partial y^2}) &= N(u_0,\frac{\partial^2 u_0}{\partial x^2},\frac{\partial^2 u_0}{\partial y^2}) \\ &+\sum_{j=1}^{\infty}N(\sum_{k=0}^{j}u_k,\sum_{k=0}^{j}\frac{\partial^2 u_k}{\partial x^2},\sum_{k=0}^{j}\frac{\partial^2 u_k}{\partial y^2}) \\ &-\sum_{j=1}^{\infty}N(\sum_{k=0}^{j-1}u_k,\sum_{k=0}^{j-1}\frac{\partial^2 u_k}{\partial x^2},\sum_{k=0}^{j-1}\frac{\partial^2 u_k}{\partial y^2}) \end{aligned} \qquad (8)$$

Substituting (Eq.(7)) and (Eq.(8)) into (Eq.(6))introduce:

$$\begin{aligned} \sum_{n=0}^{\infty}u_n = L^{-1}\{\sum_{k=0}^{m-1}s^{-1-k}u^{(k)}(x,y,0)\} &+N(u_0,\frac{\partial^2 u_0}{\partial x^2},\frac{\partial^2 u_0}{\partial y^2}) \\ &+\sum_{j=1}^{\infty}[N(\sum_{k=0}^{j}u_k,\sum_{k=0}^{j}\frac{\partial^2 u_k}{\partial x^2},\sum_{k=0}^{j}\frac{\partial^2 u_k}{\partial y^2}) \\ &-N(\sum_{k=0}^{j-1}u_k,\sum_{k=0}^{j-1}\frac{\partial^2 u_k}{\partial x^2},\sum_{k=0}^{j-1}\frac{\partial^2 u_k}{\partial y^2})] \end{aligned} \qquad (9)$$

The above equations gives results follows:

$$\begin{aligned} u_0 &= L^{-1}\{\sum_{k=0}^{m-1}s^{-1-k}u^{(k)}(x,y,0)\} \\ u_1 &= N(u_0,\frac{\partial^2 u_0}{\partial x^2},\frac{\partial^2 u_0}{\partial y^2}) \\ u_m+1 &= N(\sum_{k=0}^{m}u_k,\sum_{k=0}^{m}\frac{\partial^2 u_k}{\partial x^2},\sum_{k=0}^{m}\frac{\partial^2 u_k}{\partial y^2})-N(\sum_{k=0}^{m-1}u_k,\sum_{k=0}^{m-1}\frac{\partial^2 u_k}{\partial x^2},\sum_{k=0}^{m-1}\frac{\partial^2 u_k}{\partial y^2}) \end{aligned} \qquad (10)$$

Then the m-term numerical solution of (Eq.(1))-(Eq.(2)) is given by:

$$u(x,y,t)\cong u_0(x,y,t)+u_1(x,y,t)+.......+u_m(x,y,t), m=1,2,.... \qquad (11)$$

III. Numerical Solution of Two â€“Dimensional Diffusion Equations with Nonlocal Boundary Conditions

In this section ,the iterative Laplace transform method will be applied to solve three problems of two â€“dimensional diffusion equations with initial conditions and nonlocal boundary conditions.

Example(1):[1],[7]
consider the diffusion equation in two space variables, that is given by the following:

$$\frac{\partial u}{\partial t}=\alpha\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right);\ 0<x,y<1,\ t>0 \qquad (12)$$

where u = u(x,y,t), with Dirichlet time-dependent boundary conditions on the boundary $\partial\Omega$ of the square Ω introduce by

$$\begin{aligned} &u(0,y,t)=e^{(y+2t)}, \quad 0\le t\le T, 0\le y\le 1,\\ &u(1,y,t)=e^{(1+y+2t)}, \quad 0\le t\le T, 0\le y\le 1,\\ &u(x,0,t)=e^{(x+2t)}, \quad 0\le t\le T, 0\le x\le 1,\\ &u(x,1,t)=e^{(1+x+2t)}, \quad 0\le t\le T, 0\le x\le 1, \end{aligned} \qquad (13)$$

and nonlocal boundary condition

$$\int_0^1\int_0^1 u(x,y,t)\,dxdy=(e-1)^2 e^{2t} \qquad (14)$$

with initial conditions

$$u(x,y,0)=e^{(x+y)} \qquad (15)$$

where the exact solution is given by $u(x,y,t)=e^{(x+y+2t)}$

Taking Laplace transform on both sides of (Eq.(12))gives:

$$s\mathrm{L}\{u(x,y,t)\}-u(x,y,0)=\mathrm{L}\{\alpha(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})\} \qquad (16)$$

$$\mathrm{L}\{u(x,y,t)\}=\frac{e^{(x+y)}}{s}+\frac{1}{s}\mathrm{L}\{\alpha(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})\} \qquad (17)$$

Now,taking Laplace inverse of both sides of (Eq.(17))

$$u(x,y,t)=e^{(x+y)}+\mathrm{L}^{-1}\{\frac{1}{s}\mathrm{L}\{\alpha(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})\}\} \qquad (18)$$

Substituting (Eq.(7))and (Eq.(8)) into (Eq.(18)) and applying (Eq.(10)), obtain the components of the solution as follows:

$$\begin{aligned} &u_0(x,y,t)=e^{(x+y)}\\ &u_1(x,y,t)=2te^{(x+y)}\\ &u_2(x,y,t)=\frac{(2t)^2}{2!}e^{(x+y)}\\ &u_3(x,y,t)=\frac{(2t)^3}{3!}e^{(x+y)}\\ &\ldots\ldots\ldots\ldots \end{aligned} \qquad (19)$$

Therefore, the solution of(Eq.(12)),(Eq.(15))in a closed form can be obtained as follows:

$$u(x,y,t)=e^{(x+y)}+2te^{(x+y)}+\frac{(2t)^2}{2!}e^{(x+y)}+\frac{(2t)^3}{3!}e^{(x+y)}+\ldots+ \qquad (20)$$

Table 1: comparsion between numerical results of pade method and ILTM,t=1

x	y	solution ILTM	solution pade method	exact solution
0.0	0.0	7.38905610	7.38905610	7.38905610
0.1	0.1	9.03997223	9.04041689	9.02501350
0.2	0.2	11.06097532	11.06951484	11.02317638
0.3	0.3	13.50449822	13.54531347	13.46373804
0.4	0.4	16.46784350	16.55780082	16.44464677
0.5	0.5	20.18956786	20.21997846	20.08553692
0.6	0.6	24.60456722	24.67258833	24.53253020
0.7	0.7	29.90728645	30.09034598	29.96410005
0.8	0.8	36.60777324	36.69004490	36.59823444
0.9	0.9	44.70562963	44.74237856	44.70118449
1.0	1.0	54.59815003	54.59815003	54.59815003

Example(2):[7]
Consider the following two-dimensional diffusion problem

$$\frac{\partial u}{\partial t}=\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right);\ 0<x,y<1,\ t>0 \quad (21)$$

with initial condition

$$u(x,y,0)=(1-y)e^{x} \quad (22)$$

and the boundary conditions is given by:

$$\begin{aligned} &u(0,y,t)=(1-y)e^{t},\quad 0\le t\le 1, 0\le y\le 1,\\ &u(1,y,t)=(1-y)e^{1+t},\quad 0\le t\le 1, 0\le y\le 1,\\ &u(x,0,t)=e^{x+t},\quad 0\le t\le 1, 0\le x\le 1, \quad (23)\\ &u(x,1,t)=0,\quad 0\le t\le 1, 0\le x\le 1, \end{aligned}$$

and nonlocal boundary condition

$$\int_0^1\int_0^{x(1-x)} u(x,y,t)dxdy=2(11-4e)e^{t}, 0\le x\le 1, 0\le y\le 1, \quad (24)$$

The exact solution is given by $u(x,y,t)=(1-y)e^{x+t}$ Taking Laplace transform on both sides of (Eq.(21))introduces:

$$s\mathrm{L}\{u(x,y,t)\}-u(x,y,0)=\mathrm{L}\{(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})\} \quad (25)$$

$$\mathrm{L}\{u(x,y,t)\}=\frac{u(x,y,0)}{s}+\frac{1}{s}\mathrm{L}\{(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})\} \quad (26)$$

Now,taking Laplace inverse on both sides of (Eq.(26))observe the following Laplace equation:

$$u(x,y,t)=(1-y)e^{x}+\mathrm{L}^{-1}\{\frac{1}{s}\mathrm{L}\{(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})\}\} \quad (27)$$

The results gives the following algorithm:

$$
\begin{aligned}
&u_0(x,y,t)=u(x,y,0)=(1-y)e^x \\
&u_1(x,y,t)=t(1-y)e^x \\
&u_2(x,y,t)=\frac{t^2}{2!}(1-y)e^x \qquad (28)\\
&u_2(x,y,t)=\frac{t^3}{3!}(1-y)e^x \\
&\dots\dots\dots\dots
\end{aligned}
$$

The solution in series form is then introduce by:

$$u(x,y,t)=(1-y)e^x+t(1-y)e^x+\frac{t^2}{2!}(1-y)e^x+\frac{t^3}{3!}(1-y)e^x+\ldots+ \qquad (29)$$

Table 2: Numerical results of Example (2),t=1

x	y	solution ILTM	solution pade method	exact solution
0.0	0.0	2.71828183	2.71828183	2.71828183
0.1	0.1	2.65976522	2.63778350	2.70374942
0.2	0.2	2.60852741	2.59254212	2.65609354
0.3	0.3	2.53923642	2.50819548	2.56850767
0.4	0.4	2.40752433	2.37679262	2.43311998
0.5	0.5	2.22564212	2.18935112	2.24084454
0.6	0.6	1.95226422	1.93566543	1.98121297
0.7	0.7	1.62857433	1.60411454	1.64218422
0.8	0.8	1.19965833	1.18144246	1.20992949
0.9	0.9	0.65834215	0.65250663	0.66858944
1.0	1.0	0.00000000	0.00000000	0.00000000

Example(3):[3]

Consider the two-dimensional nonhomogeneous diffusion problem

$$\frac{\partial u}{\partial t}=\alpha\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-e^{-t}(x^2+y^2+4)\right);\ 0<x,y<1,t>0 \qquad (30)$$

The problem has initial condition

$$u(0,x,y)=1+x^2+y^2 \quad (31)$$

and the boundary conditions

$$
\begin{aligned}
&u(0,y,t)=1+y^2e^{-t}, && 0\le t\le 1, 0\le y\le 1\\
&u(1,y,t)=1+(1+y^2)e^{-t}, && 0\le t\le 1, 0\le y\le 1\\
&u(x,0,t)=1+x^2e^{-t}, && 0\le t\le 1, 0\le x\le 1 \qquad (32)\\
&u(x,1,t)=1+(1+x^2)e^{-t}, && 0\le t\le 1, 0\le x\le 1
\end{aligned}
$$

and nonlocal boundary condition

$$\int_0^1\int_0^1 u(x,y,t)dxdy=1+\frac{2}{3}e^{-t},\ 0\le x\le 1, 0\le y\le 1 \quad (33)$$

The exact solution is given by $u(x,y,t)=1+e^{-t}(x^2+y^2)$. By using iterative Laplace transform method , taking Laplace inverse on both sides of (Eq.(30)) gives:

$$s\mathrm{L}\{u(x,y,t)\}-u(x,y,0)=\mathrm{L}\{\alpha(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-e^{-t}(x^2+y^2+4))\} \qquad (34)$$

$$\mathrm{L}\{u(x,y,t)\}=\frac{u(x,y,0)}{s}+\frac{1}{s}\mathrm{L}\{\alpha(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-e^{-t}(x^2+y^2+4))\} \quad (35)$$

Now,taking Laplace inverse of both sides of (Eq.(35))

$$u(x,y,t) = 1 + x^2 + y^2 + L^{-1}\{\frac{1}{s}L\{\alpha(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - e^{-t}(x^2 + y^2 + 4))\}\} \quad (36)$$

Then above equation(Eq.(36))given the following series

$$\begin{aligned} &u_0(x,y,t) = 1 + x^2 + y^2 \\ &u_1(x,y,t) = 1 - t(x^2 + y^2) \\ &u_2(x,y,t) = 1 + \frac{t^2}{2!}(x^2 + y^2) \\ &u_3(x,y,t) = 1 - \frac{t^3}{3!}(x^2 + y^2) \\ &\ldots\ldots\ldots \end{aligned} \quad (37)$$

Table 3: Exact and numerical solution of t=1

x	y	solution ILTM	solution pade method	exact solution
0.0	0.0	1.00000000	1.00000000	1.00000000
0.1	0.1	1.00734322	1.00738402	1.00735759
0.2	0.2	1.02942120	1.02953533	1.02943036
0.3	0.3	1.06620942	1.06645182	1.06621830
0.4	0.4	1.11752532	1.11813036	1.11772142
0.5	0.5	1.18389921	1.18456735	1.18393972
0.6	0.6	1.26454422	1.26575934	1.26487320
0.7	0.7	1.36073259	1.36170370	1.36052185
0.8	0.8	1.47079257	1.47239930	1.47088568
0.9	0.9	1.59588723	1.59784805	1.59596469
1.0	1.0	1.73575888	1.73575888	1.73575888

IV. Conclusions

In this paper, introduce iterative laplace transform numerical schemes and implementation of these schemes on two dimensional diffusion equations with nonlocal boundary conditions on four boundaries.This method is conceder successfully applied to solve two dimensional diffusion equations and reduces the computational work to Largely, also It is the method can be applied to solve other nonlinear problems of partial differential equation.

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