Slightly $S^*_g$-continuous functions and totally $S^*_g$-continuous functions in Topological spaces

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Abstract: The aim of this paper is to introduce two new classes of functions, namely slightly $S^*_g$-continuous functions and totally $S^*_g$-continuous functions and study their properties.

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1. Introduction

Continuous functions in topology found a valuable place in the applications of mathematics as it has applications to engineering especially to digital signal processing and neural networks. Topologists studied weaker and stronger forms of continuous functions in topology using the sets stronger and weaker than open and closed sets.

In 1997, Slightly continuity was introduced by Jain[3] and has been applied for semi-open and pre open sets by Nour[5] and Baker[1] respectively. Recently, S.Pious Missier and J.Arul Jesti have introduced the concept of $S^*_g$-open sets[6], and introduced some more functions in $S^*_g$-open sets. Continuing this work we shall introduce a new functions called slightly $S^*_g$-continuous functions and totally $S^*_g$-continuous functions and investigated their properties in terms of composition and restriction. Also we establish the relationship between slightly $S^*_g$-continuous functions and totally $S^*_g$-continuous functions with other functions.

II. Preliminaries

Throughout this paper $(X, \tau)$, $(Y, \sigma)$ and $(Z, \eta)$ (or $X, Y$ and $Z$) represent non-empty topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset $A$ of a space $(X, \tau)$, $S^*_gCl(A)$ and $S^*_gInt(A)$ denote the $S^*_g$-closure and the $S^*_g$-interior of $A$ respectively.

Definition 2.1: A subset $A$ of a topological space $(X, \tau)$ is called a $S^*_g$-open set [6] if there is an open set $U$ in $X$ such that $U \subseteq A \subseteq S^*_gCl(U)$. The collection of all $S^*_g$-open sets in $(X, \tau)$ is denoted by $S^*_gO(X, \tau)$.

Definition 2.2: A subset $A$ of a topological space $(X, \tau)$ is called a $S^*_g$-closed set[6] if $X \setminus A$ is $S^*_g$-open. The collection of all $S^*_g$-closed sets in $(X, \tau)$ is denoted by $S^*_gCl(X, \tau)$.

Theorem 2.3 [6]: Every open set is $S^*_g$-open and every closed set is $S^*_g$-closed set.

Definition 2.4: A topological space $(X, \tau)$ is said to be $S^*_g$-$T_{1/2}$ space[7] if every $S^*_g$-open set of $X$ is open in $X$.

Definition 2.5: A topological space $(X, \tau)$ is said to be $S^*_g$-locally indiscrete space[8] if every $S^*_g$-open set of $X$ is closed in $X$.

Definition 2.6: A function $f:X \rightarrow Y$ is said to be contra-$S^*_g$-continuous [8] if the inverse image of every open set in $Y$ is $S^*_g$-closed in $X$.

Definition 2.7: A function $f:(X, \tau) \rightarrow (Y, \sigma)$ is called a contra-continuous [2] if $f^{-1}(O)$ is closed in $(X, \tau)$ for every open subset $O$ of $(Y, \sigma)$.

Definition 2.8: A mapping $f:X \rightarrow Y$ is said to be $S^*_g$-continuous [7] if the inverse image of every open set in $Y$ is $S^*_g$-open in $X$.

Definition 2.9: A map $f:X \rightarrow Y$ is said to be $S^*_g$-irresolute [7] if the inverse image of every $S^*_g$-open set in $Y$ is $S^*_g$-open in $X$.

Definition 2.10: A mapping $f:X \rightarrow Y$ is said to be strongly $S^*_g$-continuous [7] if the inverse image of every $S^*_g$-open set in $Y$ is open in $X$.

Definition 2.11: A mapping $f:X \rightarrow Y$ is said to be perfectly $S^*_g$-continuous [7] if the inverse image of every $S^*_g$-open set in $Y$ is open and closed in $X$.
Definition 2.12: A function \( f: X \to Y \) is called **slightly continuous** [3] if the inverse image of every clopen set in \( Y \) is open in \( X \).

### III. Slightly \( S^*_g \)-continuous function

**Definition 3.1:** A function \( f:(X,\tau) \to (Y,\sigma) \) is said to be **slightly \( S^*_g \)-continuous** at a point \( x \in X \) if for each subset \( V \) of \( Y \) containing \( f(x) \) there exists a \( S^*_g \)-open subset \( U \) in \( X \) containing \( x \) such that \( f(U) \subseteq V \). The function \( f \) is said to be slightly \( S^*_g \)-continuous if \( f \) is slightly \( S^*_g \)-continuous at each of its points.

**Definition 3.2:** A function \( f:(X,\tau) \to (Y,\sigma) \) is said to be **slightly \( S^*_g \)-continuous** if the inverse image of every clopen set in \( Y \) is \( S^*_g \)-open in \( X \).

**Example 3.3:** Let \( X = \{a, b, c\} = Y, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, c\}, \{a, b\}\}, \sigma = \{\emptyset, Y, \{a\}, \{b, c\}\} \). \( S^*_g(\tau) = \emptyset, X, \{a\}, \{b, c\}, \{a, b, c\} \). The function \( f \) is defined as \( f(a) = c, f(b) = a, f(c) = b \). The function \( f \) is slightly \( S^*_g \)-continuous.

**Proposition 3.4:** The definition 3.1 and 3.2 are equivalent.

**Proof:** Suppose the definition 3.1 holds. Let \( V \) be a clopen set in \( Y \) and let \( x \in f^{-1}(V) \). Then \( f(x) \in V \) and therefore there exists a \( S^*_g \)-open set \( U_x \) such that \( x \in U_x \) and \( f(U_x) \subseteq V \). Now \( x \in U_x \subseteq f^{-1}(V) \). And \( f^{-1}(V) = \cup_{x \in f^{-1}(V)} U_x \). Since arbitrary union of \( S^*_g \)-open sets is \( S^*_g \)-open, \( f^{-1}(V) \) is \( S^*_g \)-open in \( X \) and therefore \( f \) is slightly \( S^*_g \)-continuous.

Suppose the definition 3.2 holds. Let \( f(x) \in V \) where \( V \) is a clopen set in \( Y \). Since \( f \) is slightly \( S^*_g \)-continuous, \( x \in f^{-1}(V) \) where \( f^{-1}(V) \) is \( S^*_g \)-open in \( X \). Let \( U = f^{-1}(V) \). Then \( U \) is \( S^*_g \)-open in \( X \), \( x \in U \) and \( f(U) \subseteq V \).

**Theorem 3.5:** Let \( f:(X,\tau) \to (Y,\sigma) \) be a function then the following are equivalent.

1. \( f(1) \) is slightly \( S^*_g \)-continuous.
2. The inverse image of every clopen set \( V \) of \( Y \) is \( S^*_g \)-open in \( X \).
3. The inverse image of every clopen set \( V \) of \( Y \) is \( S^*_g \)-closed in \( X \).
4. The inverse image of every clopen set \( V \) of \( Y \) is \( S^*_g \)-clopen in \( X \).

**Proof:**

1. \( \Rightarrow \) (2): Follows from the Theorem 3.4.
2. \( \Rightarrow \) (3): Let \( V \) be a clopen set in \( Y \) which implies \( V^c \) is clopen in \( Y \). By (2), \( f^{-1}(V)^c = (f^{-1}(V))^c \) is \( S^*_g \)-open in \( X \). Therefore \( f^{-1}(V) \) is \( S^*_g \)-closed in \( X \).
3. \( \Rightarrow \) (4): By (2) and (3) \( f^{-1}(V) \) is \( S^*_g \)-clopen in \( X \).
4. \( \Rightarrow \) (1): Let \( V \) be a clopen subset of \( Y \) containing \( f(x) \). By (4) \( f^{-1}(V) \) is \( S^*_g \)-clopen in \( X \). Put \( U = f^{-1}(V) \) then \( f(U) \subseteq V \). Hence \( f \) is slightly \( S^*_g \)-continuous.

**Theorem 3.6:** Every slightly continuous function is slightly \( S^*_g \)-continuous.

**Proof:** Let \( f:X \to Y \) be slightly continuous. Let \( U \) be a clopen set in \( Y \). Then \( f^{-1}(U) \) is open in \( X \). Since every open set is \( S^*_g \)-open, \( f^{-1}(U) \) is \( S^*_g \)-open. Hence \( f \) is slightly \( S^*_g \)-continuous.

**Remark 3.7:** The converse of the above theorem need not be true as can be seen from the following example

**Example 3.8:** Let \( X = \{a, b, c, d\} \) with \( \tau = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\} \). \( S^*_g(\tau) = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\} \). Let \( Y = \{p, q, r\} \) with \( \sigma = \{Y, \emptyset, \{p\}, \{q, r\}\} \). Define \( f(X, \tau) \to (Y, \sigma) \) by \( f(a) = p, f(b) = q, f(c) = f(d) = r \). Hence \( f^{-1}(\{q, r\}) \subseteq \{b, c\} \) is \( S^*_g \)-open but not open in \( X \). Thus \( f \) is slightly \( S^*_g \)-continuous but not slightly \( S^*_g \)-continuous.

**Theorem 3.9:** Every \( S^*_g \)-continuous function is slightly \( S^*_g \)-continuous.

**Proof:** Let \( f:X \to Y \) be \( S^*_g \)-continuous function. Let \( U \) be a clopen set in \( Y \). Then \( f^{-1}(U) \) is \( S^*_g \)-open in \( X \) and \( f^{-1}(U) \) is \( S^*_g \)-open in \( X \). Hence \( f \) is slightly \( S^*_g \)-continuous.

**Remark 3.10:** The converse of the above theorem need not be true as can be seen from the following example.

**Example 3.11:** Let \( X = \{a, b, c\}, Y = \{p, q\} \). \( \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \). \( S^*_g(\tau) = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\} \). Define \( f(X, \tau) \to (Y, \sigma) \) by \( f(a) = q, f(b) = f(c) = p \). The function \( f \) is slightly \( S^*_g \)-continuous but not \( S^*_g \)-continuous since \( f^{-1}(\{p\}) \subseteq \{b, c\} \) is not \( S^*_g \)-open in \( X \).

**Theorem 3.12:** If the function \( f:X \to Y \) is slightly \( S^*_g \)-continuous and \( (Y, \sigma) \) is a locally indiscrete space then \( f \) is \( S^*_g \)-continuous.

**Proof:** Let \( U \) be an open subset of \( Y \). Since \( Y \) is locally indiscrete, \( U \) is closed in \( Y \). Since \( f \) is slightly \( S^*_g \)-continuous, \( f^{-1}(U) \) is \( S^*_g \)-open in \( X \). Hence \( f \) is \( S^*_g \)-continuous.

**Theorem 3.13:** If the function \( f:X \to Y \) is slightly \( S^*_g \)-continuous and \( (X, \tau) \) is a \( S^*_g^{-} \)-space then \( f \) is slightly continuous.
Proof: Let U be a clopen subset of Y. Since f is slightly $S^*_g$-continuous, $f^{-1}(U)$ is $S^*_g$-open in X. Since X is a $S^*_g - T_{1/2}$ space, $f^{-1}(U)$ is open in X. Hence f is slightly continuous.

**Theorem 3.14:** Let f: (X, τ) → (Y, σ) and g: (Y, σ) → (Z, η) be function

(i) If f is $S^*_g$-irresolute and g is slightly $S^*_g$-continuous then gof: (X, τ) → (Z, η) is slightly $S^*_g$-continuous.

(ii) If f is $S^*_g$-irresolute and g is $S^*_g$-continuous then gof is strictly $S^*_g$-continuous.

(iii) If f is $S^*_g$-irresolute and g is slightly continuous then gof is strictly $S^*_g$-continuous.

(iv) If f is $S^*_g$-continuous and g is slightly continuous then gof is strictly $S^*_g$-continuous.

(v) If f is strongly $S^*_g$-continuous and g is slightly $S^*_g$-continuous then gof is strictly continuous.

(vi) If f is strictly $S^*_g$-continuous and g is perfectly $S^*_g$-continuous then gof is $S^*_g$-irresolute.

(vii) If f is slightly $S^*_g$-continuous and g is contra-continuous then gof is strictly $S^*_g$-continuous.

(viii) If f is $S^*_g$-irresolute and g is contra-$S^*_g$-continuous then gof is strictly $S^*_g$-continuous.

**Proof:**

(i) Let U be a clopen set in Z. Since g is slightly $S^*_g$-continuous, $g^{-1}(U)$ is $S^*_g$-open in Y. Since f is $S^*_g$-irresolute, $f^{-1}(g^{-1}(U))$ is $S^*_g$-open in X. Since (gof)$^{-1}(U) = f^{-1}(g^{-1}(U))$, gof is slightly $S^*_g$-continuous.

(ii) Let U be a clopen set in Z. Since g is $S^*_g$-continuous, $g^{-1}(U)$ is $S^*_g$-open in Y. Also since f is $S^*_g$-irresolute, $f^{-1}(g^{-1}(U))$ is $S^*_g$-open in Y. Hence gof is strictly $S^*_g$-continuous.

(iii) Let O be a clopen set in Z. Then $g^{-1}(O)$ is $S^*_g$-open in Y. Therefore $f^{-1}(g^{-1}(O))$ is $S^*_g$-open in X, since f is $S^*_g$-irresolute. Hence gof is slightly $S^*_g$-continuous.

(iv) Let U be a clopen set in Z. Then $g^{-1}(U)$ is open in Y, since g is slightly continuous. Also since f is $S^*_g$-continuous. Also since f is $S^*_g$-continuous, $f^{-1}(g^{-1}(U))$ is $S^*_g$-open in X. Hence gof is slightly $S^*_g$-continuous.

(v) Let U be a clopen set in Z. Then $g^{-1}(U)$ is $S^*_g$-open in Y, since g is slightly $S^*_g$-continuous. Also $f^{-1}(g^{-1}(U))$ is open in X, since f is strongly $S^*_g$-continuous. Therefore gof is slightly continuous.

(vi) Let U be a $S^*_g$-open in Z. Since g is perfectly $S^*_g$-continuous, $g^{-1}(U)$ is open and closed in Y. Then $f^{-1}(g^{-1}(U))$ is $S^*_g$-open in X. Since f is slightly $S^*_g$-continuous. Hence gof is $S^*_g$-irresolute.

(vii) Let U be a clopen set open in Z. Since g is contra-continuous, $g^{-1}(U)$ is open and closed in Y. Since f is slightly $S^*_g$-continuous, $f^{-1}(g^{-1}(U))$ is $S^*_g$-open in X. Since (gof)$^{-1}(U) = f^{-1}(g^{-1}(U))$, gof is slightly $S^*_g$-continuous.

(viii) Let O be a clopen set in Z. Since g is contra-$S^*_g$-continuous, $g^{-1}(O)$ is $S^*_g$-open and $S^*_g$-closed in Y. Therefore $f^{-1}(g^{-1}(O))$ is $S^*_g$-open and $S^*_g$-closed in X, since f is $S^*_g$-irresolute. Hence gof is slightly $S^*_g$-continuous.

**Theorem 3.15:** If f: (X, τ) → (Y, σ) is slightly $S^*_g$-continuous and A is an open subset of X then the restriction $f_{|A}$: (A, τ|A) → (Y, σ) is slightly $S^*_g$-continuous.

**Proof:** Let V be a clopen subset of Y. Then $(f_{|A})^{-1}(V) = f^{-1}(V) ∩ A$. Since f is $S^*_g$-open and A is open, $(f_{|A})^{-1}(V)$ is $S^*_g$-open in the relative topology of A. Hence $(f_{|A})$ is slightly $S^*_g$-continuous.

IV. **Totally $S^*_g$-continuous function**

**Definition 4.1:** A map f: (X, τ) → (Y, σ) is said to be totally $S^*_g$-continuous if the inverse image of every open set in (Y, σ) is $S^*_g$-clopen in (X, τ).

**Example 4.2:** Let X = Y = [a, b, c] with τ = {X, ∅, [a], [b, c]} = $S^*_gO(X, τ)$ and σ = {Y, ∅, [a]} Define f: (X, τ) → (Y, σ) by f(a) = a, f(b) = b, f(c) = c. Then f is totally $S^*_g$-continuous.

**Theorem 4.3:** Every perfectly $S^*_g$-continuous function is totally $S^*_g$-continuous.

**Proof:** Let f: (X, τ) → (Y, σ) be a perfectly $S^*_g$-continuous function. Let U be an open set in (Y, σ). Then U is $S^*_g$-open in Y. Since f is perfectly $S^*_g$-continuous, $f^{-1}(U)$ is both open and closed in (X, τ) which implies $f^{-1}(U)$ is both open and closed in (X, τ) which implies $f^{-1}(U)$ is both $S^*_g$-open and $S^*_g$-closed in (X, τ). Hence f is totally $S^*_g$-continuous.

**Remark 4.4:** The converse of the above theorem need not be true as can be seen from the following example.

**Example 4.5:** Let X = [a, b, c] with τ = {∅, X, [a], [b, c]} = τ and Y = [d, e, f] with σ = {∅, Y, [d]}. Define f: (X, τ) → (Y, σ) by f(a) = d, f(b) = f, and f(c) = e. Here $S^*_gO(X, τ) = τ$ and $S^*_gO(Y, σ) = ∅, Y, [d], [d, e], [d, f]$). Here [d, e] and [d, f] are $S^*_g$-open in Y, but $f^{-1}([d, e]) = [a, c]$ and $f^{-1}([d, f]) = [a, b]$ are not open as well as not closed in X. So f is not perfectly-$S^*_g$-continuous but totally $S^*_g$-continuous.

**Theorem 4.6:** Every totally $S^*_g$-continuous function is $S^*_g$-continuous.

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Proof: Suppose \( f : (X, \tau) \to (Y, \sigma) \) is totally \( S^*_g \)-continuous and \( A \) is any open set in \((Y, \sigma)\). Since \( f \) is totally \( S^*_g \)-continuous, \( f^{-1}(A) \) is \( S^*_g \)-clopen in \((X, \tau)\). Hence \( f \) is \( S^*_g \)-continuous function.

Remark 4.7: The converse of the above theorem need not be true as can be seen from the following example

Example 4.8: Let \( X = \{a, b, c, d\} \) with \( \tau = \{\emptyset, X, \{a\}\} \) and \( \sigma = \{\emptyset, Y, \{a\}\} \). Here \( S^*_gO(X, \tau) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}\} \). The identity map \( f : (X, \tau) \to (Y, \sigma) \) is \( S^*_g \)-continuous but not totally \( S^*_g \)-clopen since \( f^{-1}\{a\} = \{a\} \) is \( S^*_g \)-open in \((X, \tau)\) but not \( S^*_g \)-closed in \((X, \tau)\).

Remark 4.9: The following two examples shows that totally \( S^*_g \)-continuous and strongly \( S^*_g \)-continuous are independent.

Example 4.10: Let \( X = \{a, b, c\} \) with \( \tau = \{\emptyset, X, \{a\}, \{b, c\}\} = \tau^c \) and \( Y = \{d, e, f\} \) with \( \sigma = \{\emptyset, Y, \{d\}\} \). Define \( f : (X, \tau) \to (Y, \sigma) \) by \( f(a) = d, f(b) = f \) and \( f(c) = e \). Here \( S^*_gO(X, \tau) = \{\emptyset, Y, \{d\}, \{d, e\}, \{d, f\}\} \). Here \( \{d, e\} \) and \( \{d, f\} \) are \( S^*_g \)-open in \( Y \), but \( f^{-1}\{d, e\} = \{a, c\} \) and \( f^{-1}\{d, f\} = \{a, b\} \) are not open in \( X \). Hence \( f \) is not strongly \( S^*_g \)-continuous but totally \( S^*_g \)-continuous.

Example 4.11: Let \( X = Y = \{a, b, c\} \) with \( \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} = S^*_gO(X, \tau) \) and \( \sigma = \{\emptyset, Y, \{a, b\}\} = S^*_gO(Y, \sigma) \). The identity map \( f : (X, \tau) \to (Y, \sigma) \) is strongly \( S^*_g \)-continuous but not totally \( S^*_g \)-continuous. For, the subset \( \{a, b\} \) of \((Y, \sigma)\), \( f^{-1}\{a, b\} = \{a, b\} \) is \( S^*_g \)-open in \((X, \tau)\) but not \( S^*_g \)-closed in \((X, \tau)\).

Theorem 4.12: Let \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (Y, \sigma) \to (Z, \eta) \) be function. Then \( g \circ f: (X, \tau) \to (Z, \eta) \)

(i) If\( f \) is \( S^*_g \)-irresolute and \( g \) is totally \( S^*_g \)-continuous then \( g \circ f \) is totally \( S^*_g \)-continuous.

(ii) If \( f \) is totally \( S^*_g \)-continuous and \( g \) is continuous then \( g \circ f \) is totally \( S^*_g \)-continuous.

Proof:

(i) Let \( U \) be a open set in \( Z \). Since \( g \) is totally \( S^*_g \)-continuous, \( g^{-1}(U) \) is \( S^*_g \)-clopen in \( Y \). Since \( f \) is \( S^*_g \)-irresolute, \( f^{-1}(g^{-1}(U)) \) is \( S^*_g \)-open and \( S^*_g \)-closed in \( X \). Since \( (g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \), \( g \circ f \) is totally \( S^*_g \)-continuous.

(ii) Let \( U \) be a open set in \( Z \). Since \( g \) is continuous, \( g^{-1}(U) \) is open in \( Y \). Also since \( f \) is totally \( S^*_g \)-continuous, \( f^{-1}(g^{-1}(U)) \) is \( S^*_g \)-clop in \( X \). Hence \( g \circ f \) is totally \( S^*_g \)-continuous.

References