On The Effect of Perturbations, Radiation and Triaxiality on the Stability of Triangular Libration Points In the Restricted Three-Body Problem

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Abstract: This study is aimed at studying the effects of small perturbations in the coriolis (c) and the centrifugal (ε) forces, the triaxiality (σ₁, σ₂) of the bigger primary and the radiation (q) pressure force of the smaller primary on the stability of libration point in the Restricted Three-Body Problem (RTBP) in particular to study the effect of perturbations in the coriolis and the centrifugal forces on the stability of libration points in the restricted three body problem when the bigger primary is a triaxial rigid body and the smaller primary a source of radiation. The equations of motion of the restricted problem under influence of the perturbations in the coriolis and the centrifugal forces, triaxiality and radiation were established. These equations of motion are found to be affected by the aforesaid parameters. They generalize the classical equations of motion of restricted problem and those obtained by others. Five libration points were obtained three collinear points (L₁, L₂, L₃) and two triangular points (L₄, L₅). The libration points were found to be affected by the small perturbation for the range 0 ≤ μ < μc. The critical mass value μc is affected by all the aforesaid parameters in the coriolis and centrifugal forces, the triaxial nature of the bigger primary and the radiation pressure force of the smaller primary. This generalizes the equation of orbits of the classical restricted three body problem and those obtained by others.

Keywords: Classical mechanics, Motion, Perturbations, Coriolis, Triaxial, Triangular, Symoic.

I. Introduction

The Restricted Three Body Problem (RTBP), which kept mathematicians busy for over two hundred years, describes the motion of an infinitesimal body, which moves under influence of a gravitational attraction and is not influenced by the motion of the finite bodies (primaries). This problem began with Euler (1772). In connection with this lunar theories which brought about his major accomplishment in the introduction of synodic (rotating) coordinate system. This led to the discovery of the Jacobian integral by Jacobi (1836). In recent times many properties such as shape, surface area light, perturbing forces are taken into consideration in describing the motion of satellite (both artificial and natural) of the asteroid and their stability. Due to this, many authors studied the effect of coriolis and centrifugal forces on the problem.

Wintner (1941) discovered that the stability of the two equilateral points of the triangular libration point is due to the existence of the coriolis terms in the equation of motion expressed in the rotating coordinate system. Szebehely (1967a) showed that the restricted three body problem possess five libration points, three collinear points L₁, L₂, L₃ which are unstable and the two triangular points L₄, L₅ which are stable for 0 ≤ μ ≤ μc = 0.3852 (critical mass value). He (1967b) further generalized his work and studied the effect of small perturbation in the coriolis force on the stability of libration points keeping the centrifugal force constant. He discovered that the collinear points remain unstable while for the stability of the triangular points he obtained a relation for the critical value of the mass parameter μc as

\[ μ_c = μ_o + \frac{16ε}{3\sqrt{69}} \]

where \( ε \) is the parameter for the coriolis force. Therefore establishing that the coriolis force is a stabilizing force. Subbarao and Sharma (1975) considered the same problem but with one of the primaries as an oblate spheroid and its equatorial plane coinciding with the plane of motion. While studying the stability of the triangular points they discovered that the coriolis force is not always a stabilizing force. Here, the centrifugal force is not kept constant and this paper is not in line with assertion that the coriolis force as a stabilizing force depends upon the fact that the centrifugal force is kept constant while changing the coriolis force. Based on
Szebehely’s work, Bhatnagar and Hallan (1978) considered the effect of perturbations in the coriolis ($\epsilon$) and the centrifugal ($\epsilon'$) force and discovered that the collinear points are not influenced by the perturbations and hence remain unstable. But for the triangular points they obtained the relation:

$$\mu_c = \mu_0 + 4 \frac{(36\epsilon - 19\epsilon')}{27\sqrt{69}}$$

and then concluded that the range of stability increases or decreases depending on whether the point ($\epsilon$, $\epsilon'$) lies on either of the two points in which the ($\epsilon$, $\epsilon'$) plane is divided by the line $36\epsilon - 19\epsilon' = 0$.

Singh and Iswhar (1984) determined the effect of small perturbations in the coriolis and the centrifugal forces on the location of equilibrium points in the restricted three body problem with variable mass. They found that the triangular points form nearly equilateral triangles with the primaries while the collinear points lie on the line joining the primaries. They (1985) also considered the same problem and found that the range of stability of the triangular points increases or decreases depending on whether the perturbation point ($\epsilon$, $\epsilon'$) lies in either of the two parts in which ($\epsilon$, $\epsilon'$) plane is divided by the line $36\epsilon - 19\epsilon' = 0$.

Authors like Brouwer (1946), Osipov (1970), Nikoleave (1970), Hitzl and Breakwell (1971) considered the unusual shape of the primaries or infinitesimal mass in the generalization of RTBP. They studied the effect of oblateness and triaxiality in the potential between the bodies.

Vidyakin (1974) determined the locations of five equilibrium solutions and studied their stability in the Lyapunov sense; when the primaries are ablate spheroids with the equatorial plane coinciding with the plane of motion. Furthermore Sharma and Subbarao (1979), Sharma (1982) studied the concept of restricted problem when either one of the primaries is oblate spheroid. Sharma et al. (2001) studied the stationary solutions of the planar restricted three body problem when the primaries are triaxial rigid bodies with one of the axes as the axis of symmetry and its equatorial plane coinciding of motion.

The effect of radiations was also studied by several authors taking either one or both primaries as source of radiation. This is due to ability of the primaries to emit radiation. Radzievski (1950) formulated the photogravitational RTBP. Simmons et al. (1985), Kumar and Choudry (1986) discovered the existence of libration points for the generalized photogravitational RTBP. Sharma et. al (2001) generalize their work which dealt with the stationary solution of the planar restricted three body problem when the primaries are triaxial rigid bodies, by further considering them as sources of radiation. They discovered that the state of stability of the triangular and collinear point remain unchanged.

Due to the results of various generalisation of the classical restricted three body problem, it was possible for us to consider and study a new form of generalisation.

This paper attempt to study effect of small perturbations in the coriolis and centrifugal forces on the stability of RTBP when the bigger primary is triaxial and the smaller one a source of radiation.

Since the solar radiation pressure forces $F_p$ changes with distance by the same law of gravitational attraction $F_g$ and acts in opposite direction to it, It is possible that the force lead to reduction of the effective mass of the body. Thus the resultant force on the particle is

$$F = F_g - F_p = F_g \left(1 - \frac{F_p}{F_g}\right) = qF_g$$

where $q = 1 - \frac{F_p}{F_g}$ is the mass reduction factor and the force of the body is given by $F_p = (1 - q) F_g$ such that $0 < (1 - q) << 1$.

II. Potential Of The Body

We considered the potential of a body due to one solid body of arbitrary shape and mass distribution $M$ and placed a unit mass at point $P$ outside the body and take an elementary mass $dm$ at the point $Q$ within the body.

![Fig.1:](image-url)
On The Effect of Perturbations, Radiation and Triaxiality on the Stability of…

Let O be the origin with respect to the coordinate axes P(x,y,z) and Q(ξ,η,ζ) and the centre of mass of the body.

Where OP = r and OQ = r

Potential at P due to mass dm is

\[ dV = -\frac{G \, dm}{PQ}, \]  

where G is the gravitational constant. Therefore, the potential at P due to the whole mass M is

\[ V = -\frac{GM}{r} - \frac{G}{2r^3} \left( 2I_0 - 3I \right) \]

where \( I_0 = \int r^2 \, dm \) is the moment of inertia about the origin

\[ I = \int r^2 \sin^2 \theta \, dm \]  

is the moment of inertia about the line OP.

Since the body is triaxial \( 2I_0 = I_1 + I_2 + I_3 \)

And so

\[ V = -GM \left[ \frac{1}{r} - \frac{1}{2M r^3} \left( I_1 + I_2 + I_3 - 3I \right) \right] \]

where \( I_1, I_2, I_3 \) are the principal moments of inertia about the centre of mass.

The left part of Equation (3) is the potential due to solid sphere and the right part is the potential due to the departure of the body from the spherical shape to a triaxial one.

Now,

Let O be a fixed point in the space motion and r the relative distance between the primaries of masses \( m_1 \) and \( m_2 \) (\( m_2 < m_1 \)) moving under a mutual gravitational force. Let \( r_1, r_2 \) be their distances from the origin by Newton’s law of Gravitation, the gravitation potential between two triaxial primaries is given by

\[ V^* = -Gm_1m_2 \left[ \frac{1}{r} + \frac{1}{2m_1 r^3} \left( I_1 + I_2 + I_3 - 3I \right) + \frac{1}{2m_2 r^3} \left( I'_1 + I'_2 + I'_3 - 3I' \right) \right] \]

(McCusky 1963)

Since \( m_2 \) is not triaxial \( \left( I'_1 = I'_2 = I'_3 = I \right) \), the potential for the system is

\[ V^* = -Gm_1m_2 \left[ \frac{1}{r} + \frac{1}{2m_1 r^3} \left( I_1 + I_2 + I_3 - 3I \right) \right] \]
The mean motion from Kepler’s third law of motion is

\[ n^2 = \frac{G(m_1 + m_2)}{r^3}. \]

Using Eqn. (4) becomes

\[ n^2 = G(m_1 + m_2) \left[ \frac{1}{r^3} + \frac{3}{2m(r^3)}(I_1 + I_2 + I_3 - 3I) \right] \]

III. Equations of Motion

We further considered the rotating coordinate system (Oxyz), the x axis is taken along the line joining the triaxial body of the mass \( m_1 \) and the radiating smaller body of mass \( m_2 \) and that they move without rotation about their centre of mass in a circular orbit with an angular velocity \( n \) of the infinitesimal body with mass \( m \). Then the kinetic energy is given by

\[
\frac{1}{2} m \left[ (\dot{x}^2 + \dot{y}^2) + 2n(x\dot{y} - \dot{x}y) + n^2(x^2 + y^2) \right] = T_2 + T_1 + T_0
\]

Where

\[ T_2 = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) \]

\[ T_1 = mn(x\dot{y} - \dot{x}y) \]

\[ T_0 = \frac{1}{2} mn^2(x^2 + y^2) \]

And gravitational potential energy between the infinitesimal mass \( m \) and the triaxial bigger primary and the radiating smaller primary is

\[ V = -Gm \left[ \frac{m_1}{r_1} + \frac{m_2 q}{r_2} + \frac{m_1}{2m_1 r_1^3}(I_1 + I_2 + I_3 - 3I) \right] \]

Where

\[ r_1^2 = (x - a)^2 + y^2 \]

\[ r_2^2 = (x - b)^2 + y^2 \]

\( I_1, I_2, I_3 \) are the principal moments of inertia of the triaxial rigid body of mass \( m_1 \) at its centre of mass with \( a, b, c \), as lengths of its semi-axes.

\( I = I_1 + I_2 + I_3 \) is the moment of inertia about a line joining the centre of \( m_1 \) and \( m_2 \) where \( l, m, \) are the direction cosines of the line with respect to its principal axes.

q is the mass reduction factor due to radiation.

Adapting the Hamiltonian canonical equation of motion expression

\[ \dot{x} = \frac{\partial H}{\partial p_x}, \dot{y} = \frac{\partial H}{\partial p_y}, \dot{p}_x = -\frac{\partial H}{\partial x} \quad \text{and} \quad \dot{p}_y = -\frac{\partial H}{\partial y} \]

with

\[ H = T_2 - T_0 + \frac{1}{2m} \left[ p_x^2 + p_y^2 \right] + n(P_x y - x P_y) + U \quad \text{U} = V - T_0 \]

The equations of motion of the infinitesimal mass \( m \) are

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\[
\ddot{x} - 2n\dot{y} = \frac{\partial U}{\partial x}
\]
\[
\ddot{y} + 2n\dot{x} = \frac{\partial U}{\partial y}
\]
where
\[
U = -Gm \left[ \frac{m_1}{r_1} + \frac{m_2q}{r_2} + \frac{m_1}{2m_1r_1^3} \left( I_1 + I_2 + I_3 - 3I \right) \right]
\]
\[
r_1^2 = (x - a)^2 + y^2
\]
\[
r_2^2 = (x - b)^2 + y^2
\]

we choose the sum of the masses of the primaries to be one so that if \( m_2 = \mu, m_1 = 1 - \mu \), where \( \mu \) is the ratio of the mass of the smaller primary to the total mass of the primaries \( \mu = \frac{m_2}{m_1 + m_2} \leq \frac{1}{2} \) and \( 0 < \mu \leq \frac{1}{2} \). The distance between the primaries is taken equal to one and the unit of time is so chosen as to make the gravitational constant \( G \) is unity. We have the origin as the barycentre of the masses \( m_1 \) and \( m_2 \) as defined by Szebehely (1967b). The equations of motion in the dimensionless synodic coordinate system \((Oxyz)\) become

\[
\ddot{x} - 2n\dot{y} = n^2 x - \frac{\partial \overline{U}}{\partial x}
\]
\[
\ddot{y} + 2n\dot{x} = n^2 y - \frac{\partial \overline{U}}{\partial y}
\]
Where
\[
\overline{U} = \frac{1 - \mu}{r_1} + \frac{\mu q}{r_2} + \frac{1 - \mu}{2m_1r_1^3} \left( I_1 + I_2 + I_3 - 3I \right)
\]

\[
r_1^2 = (x - \mu)^2 + y^2
\]
\[
r_2^2 = (x + 1 - \mu)^2 + y^2
\]

The mean motion \( n \) is
\[
n^2 = 1 + \frac{3}{2} \left( 2A_1 - A_2 - A_3 \right).
\]
\[
I_1 = \frac{m_1}{5} (b^2 + c^2), \quad I_2 = \frac{m_1}{5} (a^2 + c^2), \quad I_3 = \frac{m_1}{5} (b^2 + a^2).
\]
and
\[
A_1 = \frac{a^2}{5R^2}, \quad A_2 = \frac{b^2}{5R^2}, \quad A_3 = \frac{c^2}{5R^2},
\]
where \( R \) is the dimensional distance between the primaries. Neglecting the drag effect,

Introducing small perturbations \( \varepsilon \) and \( \varepsilon' \) in the coriolis and the centrifugal forces using the parameter \( \phi \) and \( \psi \), respectively, such that
\[
\phi = 1 + \varepsilon \quad |\varepsilon| << 1
\]
\[
\psi = 1 + \varepsilon' \quad |\varepsilon'| << 1
\]
the equations of motion in (13) and (14) become
\[ \ddot{x} - 2\phi \dot{y} = \frac{\partial \Omega}{\partial x} \]
\[ \ddot{y} + 2\phi \dot{x} = \frac{\partial \Omega}{\partial y} \]  
(17)

Where 
\[ \Omega = \frac{n^2}{2} \left( x^2 + y^2 \right) + \frac{1 - \mu}{r_1} + \frac{\mu q}{r_2} + \frac{1 - \mu}{2m_r r_1^3} \left( I_1 + I_2 + I_3 - 3I \right) \]

or
\[ \Omega = \frac{n^2}{2} \left( x^2 + y^2 \right) + \frac{1 - \mu}{r_1} + \frac{\mu q}{r_2} + \frac{(1 - \mu)}{2r_1^3} (2\sigma_1 - \sigma_2) - \frac{3(1 - \mu)(\sigma_1 - \sigma_2)}{2r_1^5} y^2 \]  
(18)

Where \( \sigma_1 = A_1 - A_3, \sigma_2 = A_2 - A_3 \), and \( \sigma_1, \sigma_2 << 1 \),

the mean motion, in Eqn.(16) become
\[ n^2 = 1 + \frac{3}{2} (2\sigma_1 - \sigma_2) \]  
(19)

IV. Location of Libration Points

Equation (17) admits the Jacobi integral
\[ x^2 + y^2 - 2\Omega + C = 0 \]

The libration points are the singularities of the manifold
\[ F(x, y, \dot{x}, \dot{y}) = x^2 + y^2 - 2\Omega + C = 0 \]

These points are the solutions of the equations.
\[ \Omega_x = 0, \quad \Omega_y = 0 \]

That is
\[ \Omega_x = n^2 \psi x - \frac{(1 - \mu)}{r_1^3} \left( x - \mu \right) - \frac{\mu q}{r_1^3} (x + 1 - \mu) - \frac{3(1 - \mu)}{2r_1^5} (x - \mu)(2\sigma_1 - \sigma_2) \]
\[ + \frac{15(1 - \mu)}{2r_1^7} (x - \mu)(\sigma_1 - \sigma_2) y^2 = 0 \]  
(20)

\[ \Omega_y = \left[ n^2 \psi - \frac{1 - \mu}{r_1^3} - \frac{\mu q}{r_2^3} - \frac{3(1 - \mu)(4\sigma_1 - 3\sigma_2)}{2r_1^5} + \frac{15(1 - \mu)(\sigma_1 - \sigma_2)}{2r_1^7} y^2 \right] y = 0 \]  
(21)

Location of Triangular Points

the triangular points are the solutions of Eqns. (20) and (21) for \( y \neq 0 \) i.e. from (21)
\[ n^2 \psi - \frac{(1 - \mu)}{r_1^3} - \frac{\mu q}{r_2^3} \left( 1 - \mu \right) (2\sigma_1 - \sigma_2) + \frac{15}{2r_1^7} (1 - \mu)(\sigma_1 - \sigma_2) = 0 \]  
(22)

If the bigger primary is not triaxial (\( \sigma_1 = \sigma_2 = 0 \)) and the smaller one is not radiating (\( q = 1 \)) and \( n=1 \), Eqns. (21) and (22) becomes
\[ \left[ \psi - \frac{1 - \mu}{r_1^3} - \frac{\mu}{r_2^3} \right] x + \mu (1 - \mu) \left[ \frac{1}{r_1^3} - \frac{1}{r_2^3} \right] = 0 \]

and
\[ \left[ \psi - \frac{1 - \mu}{r_1^3} - \frac{\mu}{r_2^3} \right] = 0 \]

Solving these gives \( r_1 = r_2 = \frac{1}{\psi^{\frac{1}{3}}} \)

Now, due the triaxiality of bigger primary (\( \sigma_1 \neq \sigma_2 \neq 0 \)) and the radiation effect of the smaller one (\( q \neq 1 \)) we can assume that the solutions to Eqns (21) and (22) to be
\( r_1 = \frac{1}{\psi^{1/3}} + \alpha \), \( r_2 = \frac{1}{\psi^{2/3}} + \beta \) where \( \sigma, \beta \) are negligible (i.e. \( /a, /b, /c, /d, \) are very small), we have

\[
\alpha = -\frac{1}{128}\left(1 - \frac{8}{9}\psi^{2/3} - 5\psi^{4/3}\sigma_1 - \frac{1}{2} + 3\psi^{2/3} - 5\psi^{4/3}\sigma_2\right)
\]

\[
\beta = -\frac{1}{128}\left(\frac{1}{3}(1 - q) + 1 - \frac{2\psi^{2/3} + \psi^{2/3} + 2\psi^{2/3}}{2}\sigma_1 - \frac{1}{2} - \frac{2\psi^{2/3} + \psi^{2/3} + 2\psi^{2/3}}{2}\sigma_2\right)
\]

Then we obtain the co-ordinates of the triangular libration points \( L_4 (x_1, + y) \) and \( L_5 (x_1, - y) \) as

\[
x = \mu - \frac{1}{2\psi^{1/3}} - \frac{1}{2\psi^{2/3}} + \frac{1}{\psi^{1/3}} - \frac{1}{\psi^{2/3}} \left(\frac{1}{3}(1 - q) - \left(2\psi^{2/3} + \frac{3\psi^{2/3} + \psi^{2/3} + 2\psi^{2/3}}{2}\sigma_1 + \left(1 - \frac{2\psi^{2/3} + 2\psi^{2/3} + 3\psi^{2/3}}{2}\sigma_2\right)\right) - \frac{3}{2} - \frac{1}{\psi^{1/3}} - \frac{1}{\psi^{2/3}} \left(\frac{1}{\psi^{1/3}} - \frac{1}{\psi^{2/3}}\right)\sigma_1 - \frac{3}{\psi^{1/3}} - \frac{3}{\psi^{2/3}} \left(\frac{1}{\psi^{1/3}} - \frac{1}{\psi^{2/3}}\right)\sigma_2\right)
\]

\[
y = \pm \sqrt{\frac{4 - \psi^{1/3}}{2\psi^{1/3}}} \left(1 + \frac{2\psi^{1/3}}{\psi^{2/3}} (\alpha + \beta)\right)
\]

Substituting Eqs. (23) and (24) in (20) and (22) neglecting second and higher order terms (since \( /a, /b, /c, /d, \) are very small), we have

\[
ext = \mu - \frac{1}{2\psi^{1/3}} - \frac{1}{2\psi^{2/3}} + \frac{1}{\psi^{1/3}} - \frac{1}{\psi^{2/3}} \left(\frac{1}{3}(1 - q) - \left(2\psi^{2/3} + \frac{3\psi^{2/3} + \psi^{2/3} + 2\psi^{2/3}}{2}\sigma_1 + \left(1 - \frac{2\psi^{2/3} + 2\psi^{2/3} + 3\psi^{2/3}}{2}\sigma_2\right)\right) - \frac{3}{2} - \frac{1}{\psi^{1/3}} - \frac{1}{\psi^{2/3}} \left(\frac{1}{\psi^{1/3}} - \frac{1}{\psi^{2/3}}\right)\sigma_1 - \frac{3}{\psi^{1/3}} - \frac{3}{\psi^{2/3}} \left(\frac{1}{\psi^{1/3}} - \frac{1}{\psi^{2/3}}\right)\sigma_2\right)
\]

(b) to the position of the triangular points \( L(x, y) \) such that \( x = a + \xi \) and \( y = b + \eta \). applying this in the equations of motion (17), we obtain the variational equation of motion (first order in terms of \( \xi, \eta \)) as

\[
\dot{\xi} - 2n\phi \eta = \Omega_{x, y}^0 \xi + \Omega_{x, y}^0 \eta
\]

\[
\dot{\eta} + 2n\phi \dot{\xi} = \Omega_{x, y}^0 \xi + \Omega_{x, y}^0 \eta
\]

The superscript \( (1) \) indicates that the derivatives are evaluated at the libration points.

And its corresponding characteristic equation is

\[
\lambda^4 - \left(\Omega_{x, y}^0 + 4n^2 \phi^2 \right) \lambda^2 + \Omega_{x, y}^0 \Omega_{y, y}^0 - \left(\Omega_{x, y}^0 \right)^2 = 0
\]

where

\[
\Omega_{x, y}^0 = \frac{3}{4} \left(1 - q\right) - \frac{3}{2} \mu \left(1 - q\right) + \left(\frac{57}{16} - \frac{3}{2\mu} + \frac{45}{16}\right) \sigma_1 + \left(-\frac{3}{16} + \frac{3}{2\mu} - \frac{93}{16} \mu\right) \sigma_2
\]

\[
\Omega_{y, y}^0 = \sqrt{\frac{3}{2} - \frac{1}{2} \mu \left(1 - q\right) + \left\{\frac{11\left(\mu - 1\right)}{6} \left(\frac{1}{2}\mu - 1\right) \left(1 - q\right) + \left(-\frac{47}{16} + \frac{1}{2\mu} + \frac{89}{16}\right) \sigma_1 + \left(-\frac{9}{16} + \frac{1}{2\mu} - \frac{37}{16} \mu\right) \sigma_2\right\}
\]

\[
\Omega_{x, y}^0 = \frac{9}{4} \left(1 - q\right) - \frac{3}{2} \mu \left(1 - q\right) + \left(\frac{87}{16} + \frac{3}{2\mu} - \frac{45}{16} \mu\right) \sigma_1 + \left(-\frac{21}{16} + \frac{3}{2\mu} + \frac{45}{16} \mu\right) \sigma_2
\]

We replace \( \lambda^2 \) in the characteristic equation (27) by \( \Lambda^2 \) and thus it becomes:

\[
\Lambda^2 - P \Lambda + Q = 0
\]

Where,

\[
P = 1 + 8 \epsilon - 3 \epsilon' + 3 \sigma_1 + \left(-\frac{9}{2} + 3\mu\right) \sigma_2 \quad \text{P>0 since } 0<\mu<1/2 \), \( |\epsilon'|, |\epsilon|, |\sigma_1| \) \quad \text{and} \quad |\sigma_2| \ll 1
\]

\[
Q = \left[\begin{array}{c}
\frac{27\mu}{4} \left(1 - \mu\right) - \frac{33\mu}{4} \left(1 - \mu\right) \epsilon' + \frac{3\mu}{2} \left(1 - \mu\right) 
\end{array}\right] \left(\begin{array}{c}
\left(-\frac{45}{8} + \frac{891}{16}\mu - \frac{801}{16} \mu^2\right) \sigma_1 
\end{array}\right)
\]

\[
+ \left(\begin{array}{c}
\frac{45}{8} - \frac{423}{16} \mu + \frac{333}{16} \mu^2
\end{array}\right) \sigma_2
\]

The roots of Eqn (28) are:

\[
\Lambda_{1,2} = \frac{1}{2} \left[ -P \pm \sqrt{P^2 - 4Q} \right]
\]

Which consequently are

\[
\Lambda_1 = +\Lambda_{1/2}, \Lambda_2 = -\Lambda_{1/2}, \Lambda_3 = +\Lambda_{2/3}, \Lambda_4 = -\Lambda_{2/3}
\]

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These roots depend on the value of the mass parameters $\mu$, the perturbations in the coriolis and the centrifugal forces, the radiation pressure of the smaller primary and the triaxial nature of the bigger primary and are controlled by the discriminant $\Delta$

From Eqn. (28)

$$\Delta = P^2 - 4Q$$

$$= 1 - 6\varepsilon + 16\varepsilon - 2\left(1 + \frac{22}{9}\varepsilon\right)\mu(1 - \mu) - 6\mu(1 - \mu)(1 - q)$$

$$= -\frac{9}{4}\left(-\frac{38}{3} + 99\mu - 89\mu^2\right)\sigma_1 - \frac{9}{4}\left(14 - \frac{147}{3}\mu + 37\mu^2\right)\sigma_2$$

If $\mu = 0$, $\Delta > 0$ and $\mu = \frac{1}{2}$, $\Delta < 0$ ( $|\varepsilon|$, $|\sigma_1|$, and $|\sigma_2| << 1$). Since the discriminant $\Delta$ are of opposite sign at different values of $\mu$, there is only one value of $\mu$ in the open interval $(0, \frac{1}{2})$ for which $\Delta$ vanishes. The value of $\mu$ is called the critical value of the mass parameter $\mu$ denoted by $\mu_c$.

### VI. Critical mass

If the bigger primary is not triaxial ($\sigma_1 = \sigma_2 = 0$) the smaller primary is not radiating ($q = 1$) and there are no perturbation effects in the coriolis and centrifugal forces ($\varepsilon = \varepsilon = 0$) and the discriminant equal zero, then the root of (28) is

$$\mu = \mu_0 = \frac{1}{2} \left[ 1 - \sqrt{\frac{23}{27}} \right] = 0.038520$$

(Szbehely1967b) (31)

Now if the bigger primary is triaxial ($\sigma_1, \sigma_2 \neq 0$), the smaller one a source of radiation ($q \neq 1$) and there are perturbations in the coriolis and centrifugal forces ($\varepsilon', \varepsilon' \neq 0$), then from setting Eqn. (30) to be equal to zero

$$\mu_c = \mu_0 + \mu_p + \mu_r + \mu_t$$

where

$$\mu_0 = \frac{1}{2} \left[ 1 - \sqrt{\frac{23}{27}} \right]$$

$$\mu_p = \frac{4(26\varepsilon + 19\varepsilon')}{27\sqrt{69}}$$

$$\mu_r = \frac{2}{27\sqrt{69}} (1 - q)$$

$$\mu_t = \frac{1}{2} \left[ \frac{5}{6} + \frac{59}{9\sqrt{69}} \right] \sigma_1 - \frac{1}{2} \left( \frac{19}{18} \sigma_1 + \frac{85}{9\sqrt{69}} \right) \sigma_2$$

Equation (32) shows the effect of perturbation in the coriolis and the centrifugal forces, $\mu_p$, the triaxial effect, $\mu_t$ from the bigger primary and the radiating effect, $\mu_r$, from the smaller one, on the critical mass value.

We considered the mass parameter $\mu$ in three region:

i) When $0 < \mu \leq \mu_c$ we have that

$$0 < \Delta \leq P^2$$

$$\Rightarrow -\frac{1}{2} \leq \frac{\Delta}{\mu^2} < 0$$

and

$$-P^2 \leq -\Delta < 0 \Rightarrow -P \leq \Lambda_2 < -\frac{1}{2} \mu$$

written as

$$\lambda_{1,2} = \pm \sqrt{-\Lambda_1} = \pm \sqrt{-(\Lambda_1)}^{1/2} = \pm is_1$$

$$\lambda_{3,4} = \pm \sqrt{-\Lambda_2} = \pm \sqrt{-(\Lambda_2)}^{1/2} = \pm is_2$$

(32)

Where from Eqn.(29) and for very small value of $\mu$, ($\mu \approx 0$),

$$s_1 = \left[ \frac{27\mu - 45\sigma_1 + 45\sigma_2}{8\sigma_1 + 8\sigma_2} \right]^{1/2}$$

$$s_2 = \left[ 1 - \frac{27\mu + 4\varepsilon - \frac{3}{2}\varepsilon' \sigma_1 - \frac{81}{16} \sigma_2}{8\sigma_1 + 8\sigma_2} \right]$$

The roots are purely imaginary, hence the triangular point are stable in this region.

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ii) When \( \mu_c < \mu \leq \frac{1}{2} \), we have that
\[ \Delta < 0 \] and
\[ \Lambda_{1,2} = \frac{1}{2} \left[ -P + \sqrt{\Delta} \right] \]
This implies
\[ \Lambda_{1,2} = \frac{1}{2} \left[ -P + i \delta \right] \] where \( \delta = +\sqrt{\Delta} > 0 \)
\[ \geq \left( \frac{27(1 - \mu)}{\sqrt{2}} \right) \left[ 1 + \frac{11}{18} e - \frac{1}{18} (1 - q) + \frac{1}{48} \left( 89 - \frac{10}{\mu} \right) \sigma_1 - \frac{1}{48} \left( 37 - \frac{10}{\mu} \right) \sigma_2 \right] \]
Here, the roots of the characteristic equations are:
\[ \lambda_1,2 = \pm \Lambda_{1,2}^{\frac{1}{2}} \]
and \( \lambda_{3,4} = \pm \Lambda_{1,2}^{\frac{1}{2}} \) or
\[ \lambda_1 = \left[ \frac{1}{2} (-P + i \delta) \right]^{\frac{1}{2}} = \frac{1}{\sqrt{2}} \sqrt{-P + i \delta} = \alpha_1 + i \beta_1 \]
\[ \lambda_2 = \left[ \frac{1}{2} (-P - i \delta) \right]^{\frac{1}{2}} = -\frac{1}{\sqrt{2}} \sqrt{-P - i \delta} = \alpha_2 + i \beta_2 \]
\[ \lambda_3 = \left[ \frac{1}{2} (-P - i \delta) \right]^{\frac{1}{2}} = \frac{1}{\sqrt{2}} \sqrt{-P - i \delta} = \alpha_3 + i \beta_3 \]
and the lengths of the roots are which are equal is given by:
\[ |\lambda| = |\lambda_1,2,3,4| = \frac{1}{\sqrt{2}} \sqrt{P^2 + \delta^2} \]
Using Eqns. (29) and (33), we have
\[ |\lambda| = \left[ \frac{27(1 - \mu)}{\sqrt{2}} \right] \left[ 1 + \frac{11}{18} e - \frac{1}{18} (1 - q) + \frac{1}{48} \left( 89 - \frac{10}{\mu} \right) \sigma_1 - \frac{1}{48} \left( 37 - \frac{10}{\mu} \right) \sigma_2 \right] \]
Now from the root
\[ \lambda = \alpha + i \beta = \frac{1}{\sqrt{2}} \sqrt{-P + i \delta} \]
Comparing and equating the real and imaginary part, we have
\[ \alpha = \frac{\delta}{2\sqrt{P + 2|\lambda|^2}} > 0 \quad \beta = \frac{1}{2} \sqrt{P + 2|\lambda|^2} > 0 \] \quad (34)
Therefore, the principal argument of the first root is
\[ \text{arg}(\lambda) = \theta = \theta_1 = \arctan \left( \frac{\beta}{\alpha} \right) = \arctan \left( \frac{P \pm \sqrt{P^2 + \delta^2}}{\delta} \right) \]
where \( P \) and \( \delta \) are given by equations (29) and (33)
The argument of the four roots are related by
\[ \theta = \theta_1 \pi - \pi = 2\pi - \theta_0 = \pi - \theta_4 \]
The real and imaginary part of the roots \( \alpha_i \) and \( \beta_i \) are related by
\[ \alpha = \alpha_1 = -\alpha_2 = \alpha_3 = -\alpha_4 \text{ and } \beta = \beta_1 = -\beta_2 = -\beta_3 = \beta_4 \]
where \( \alpha \) and \( \beta \) are given in Eqn. (34)
It follows that in this region the triangular points are unstable since the real parts of two of the values of the characteristic roots are positive.

iii) when \( \mu = \mu_c \) we have that \( \Delta = 0 \).
Consequently

\[ \lambda_1 = \lambda_1^{(1)} = \pm i\sqrt{\frac{1}{2} P} \quad \text{and} \quad \lambda_2,4 = -\lambda_1^{(1)} = \pm i\sqrt{\frac{1}{2} P}, \text{ where } P > 0 \text{ as given.} \]

The double roots give secular terms in the solution of the variational equation of motion. Therefore, the triangular point is unstable.

**VII. Discussion**

Equations (17) and (18) describe the motion of the infinitesimal mass when the bigger primary is a triaxial rigid body and the smaller one is a source of radiation, under the influence of perturbations in the coriolis and centrifugal forces. Equation (17) depends on the parameter, \( \phi \), of the perturbation in the coriolis force, while equation (18), which is the force function, depends on the parameter \( \psi \) of the centrifugal force. Both equations depend on the triaxiality coefficients \( \sigma_1, \sigma_2 \) and the radiation factor, \( q \). Equation (19) shows that the mean motion is affected by only the triaxiality of the bigger primary.

If the bigger primary is not triaxial (\( \sigma_1 = \sigma_2 = 0 \)), the smaller primary not radiating (\( q = 1 \)) and there are no perturbations in the coriolis and centrifugal forces, then the equations of motion reduce to that of classical problem (Szebehely, 1967b). In the absence of triaxiality and radiation, the equation of motion coincides with those obtained by Bhatnagar and Hallan (1979). When there are no perturbations in the coriolis and centrifugal forces, the equations of motion reduce to that of Sharma et al. (2001). If the bigger primary is not triaxial but an oblate spheroid (\( \sigma_1 = \sigma_2 \neq 0 \)) the equations of motion agree with those of AbdulRaheem and Singh (2006).

Equations (32) show the effects of perturbations in the coriolis and the centrifugal forces on the critical value of the mass parameter. This value determines the stability of libration points in the restricted problem. In the absence of perturbation potentials, triaxiality of the bigger primary and radiation force of the smaller primary, the critical value of the mass parameter reduces to

\[ \mu_0 = \frac{1}{2} \left( 1 - \sqrt{\frac{23}{27}} \right) = 0.038520... \quad \text{(Szebehely, 1967b)} \]

which is mass ratio for the classical restricted problem. But in the absence of only perturbation potential (\( \mu_0 = 0 \)) the critical mass value \( \mu_\epsilon \) agrees with that of Sharma et al. (2001). If the bigger primary is not triaxial but oblate (\( \sigma_1 = \sigma_2 \neq 0 \)) and (\( \mu_0 = 0 \)) then it verifies the result of Singh and Ishwar (1999). In this case \( \mu_\epsilon < \mu_0 \) and this implies that the range of stability decreases.

When the bigger primary is not triaxial (\( \mu_0 = 0 \)) and the smaller one is not radiating (\( \mu_\epsilon = 0 \)) the critical mass value become

\[ \mu_\epsilon = \frac{4(36\epsilon - 19\epsilon')}{27\sqrt{69}} \quad \text{(Bhatnagar and Hallan 1979).} \]

In this case \( \mu_\epsilon > \mu_0 \) and it implies that the range of stability increases. Hence, we conclude that the perturbation potentials, the triaxiality of the bigger primary and radiation pressure force of the smaller primary have destabilizing tendency on the problem. If \( \mu_\epsilon = 0, 36\epsilon - 19\epsilon' = 0 \) we have a straight line which divides the plane \((\epsilon, \epsilon')\) into two parts: \( \pi_1 \) on the right for points \( 36\epsilon - 19\epsilon' > 0 \) while \( \pi_2 \) on the left for points \( 36\epsilon - 19\epsilon' < 0 \).

In the part \( \pi_1, \mu_\epsilon < \mu_0, (\mu_\epsilon, \mu_0 < 0) \), that is the range of stability decreases and for a point \((\epsilon, \epsilon')\) in the part \( \pi_2, \mu_\epsilon < \mu_0 \) this also implies that the range of stability decreases. For a point \((\epsilon, \epsilon')\) lying on the line \( 36\epsilon - 19\epsilon' = 0, \mu_\epsilon < \mu_0 \) again the range of stability decreases. Therefore, the effect of perturbations in the coriolis and centrifugal forces, the triaxiality of the bigger primary and radiation force the smaller primary is that they decrease the range of stability.

As a particular case, when the point \((\epsilon, \epsilon')\) coincides with the origin i.e. \( \epsilon = 0, \epsilon' = 0 \).

\[ \mu_\epsilon = \mu_0 + \mu_\epsilon + \mu_1 \quad \text{(Sharma et al. 2001)} \]

For the point lying on the \( \epsilon \) axis, \( \epsilon' = 0 \)

\[ \mu_\epsilon = \mu_0 + \mu_\epsilon + \frac{16\epsilon}{3\sqrt{69}} \]

From this, we see that for \( \epsilon > 0, \mu_\epsilon > \mu_0 \). For \( \epsilon < 0, \mu_\epsilon < \mu_0 \) and establishes that the coriolis force is a stabilizing force provided the centrifugal force is kept constant. This verifies the result of Szebehely (1967a) though the stability ability is weakened due to the triaxiality of the bigger primary and the radiation pressure force of the smaller primary.
For point lying on the $\varepsilon'$-axis, $\varepsilon = 0$

$$\mu_{\varepsilon} = \mu_1 + \mu_3 + \mu_5 - \frac{76e'}{27\sqrt{69}}$$

This shows that for $e' > 0$, $\mu_{\varepsilon} < \mu_1$ and for $e' < 0$, $\mu_{\varepsilon} > \mu_1$, implying that the centrifugal force is a destabilizing force when the coriolis force is kept constant. That is, irrespective of when $\mu_1, \mu_3$ are equal to zero. Here we see that the triaxiality of the bigger primary and the radiation pressure force of the smaller one increase the destabilizing tendency of the centrifugal force. This result agrees with those of Subbarao and Sharma (1975) and AbdulRaheem and Singh (2006) when $\sigma_1 = \sigma_2 \neq 0$.

VIII. Conclusion

Hence under the effect of perturbation in the coriolis and the centrifugal forces in the restricted three body problem when the bigger primary is truncated and the smaller one radiating the triangular points are stable for $0 \leq \mu \leq \mu_5$ and unstable for $\mu_4 \leq \mu \leq \frac{1}{2}$. This may be applied to examine the asteroids librate around the Lagrangian points in the Sun-Planets systems or satellite in the Earth-Moon system.

References


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