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Martingales Stopping Time Processes

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Abstract: *We begin with some preliminaries on measure-theoretic probability theory which in turn, allows us to discuss the definition and basic properties of martingales. We then discuss the idea of stopping times and stopping process. We state auxiliary results and prove a theorem on a stopping process using the Càdlàg property of the process.*
Keywords: *Càdlàg, martingale, stopping process, stopping time*

I. Introduction

We shall provide explanation of the theory on martingale and the formal definition. We also define a stopping time, provide some simple explanation of the classical (discrete) time and other related stopping times. The concept of stochastic process and Càdlàg property will be introduced and at last use same to prove a theorem on a stopping process.

In modern probability theory the model for a random experiment is called a probability space and this consists of the triple $(\Omega, \mathcal{F}, P)$ where Ω is a set, called the sample space, $\mathcal{F}$ is a σ-algebra of the subsets of Ω and P is a probability measure on $(\Omega, \mathcal{F})$, meaning that every set in $\mathcal{F}$ is measurable and $P(\Omega) = 1$. The measure P gives the probability that an outcome occurs. In discrete probability, one is usually interested in random variables which are real-valued functions on the sample space. A random variable X will then be a function $X : \Omega \to \mathbb{R}$ which is measurable with respect to $\mathcal{F}$. The expected value of a random variable X is its integral with respect to the measure P, i.e.

$$E(X) = \int_{\Omega} X(\mathrm{w})\, dP(\mathrm{w}),\ \ \mathrm{w} \in \Omega$$

and we say that a random variable X is integrable if $E(|X|) < \infty$. In the case of the conditional expectation, given a sub-σ-algebra A of $\mathcal{F}$, the conditional expectation $E(X \mid \mathrm{A})$ is the expectation of X given that the events contained in A have occurred. A filtration on $(\Omega, \mathcal{F}, P)$ is an increasing sequence $\{\mathcal{F}\}_{k=1}^{\infty}$ of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_3 \subset \ldots \subset \mathcal{F}$.

1.1 Martingale

It is already known that a martingale is (informally) a random process $X = \{X_k\}$ which models a player's fortune in a fair game. That is his expected fortune at time k given the history up to this point is equal to his current fortune.

$$E\left[X_{k+1} \mid X_1, \ldots, X_k\right] = X_k \tag{1.1.1}$$

This in turn implies for all k:

$$E(X_{k+1}) = E(X_k) = E(X_{k-1}) = \ldots = E(X_1) = E(X_0) \tag{1.1.2}$$

So, player's expected fortune at any time is equal to his starting expected fortune.

Basically, the requirements for a martingale are a probability space $(\Omega, \mathcal{F}, P)$, a sequence of σ-algebras $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n \subset \mathcal{F}$ and a sequence of variables $X_0, X_1, \ldots, X_n$, that is a stochastic process. Filtrations are important because they provide a concise way of finding martingale, since the conditions for a martingale are that:

(a) Each X_k is $\mathcal{F}_k$-measurable (i.e. if we know the information in $\mathcal{F}_k$, then we know the value of X_k). This is to say that $\{X_k\}$ is adapted to the filtration $\{\mathcal{F}_k\}$ and for each k,

(b) $E(|X_k|) < \infty\ \forall k$

(c) $E\left[X_{k+1} \mid \mathcal{F}_k\right] = X_k$, a.s.

This means that martingales tend to go neither up nor down, [3]. Martingales are allegory of 'life' [5]. Supermartingales tend to go down, $E\left[X_{k+1} \mid \mathcal{F}_k\right] \le X_k$ and sub-martingales tend to go up, $E\left[X_{k+1} \mid \mathcal{F}_k\right] \ge X_k$. See also, [6] and [7]. To find when to stop a martingale at random time, one needs a rule for stopping a random process which does not depend on future.

Returning to (1.1.2), one asks whether the game remains fair when stopped at a randomly chosen time. Specifically, if τ is a random stopping time and X_τ denotes the game stopped at this time, do we have $E\left(X_\tau\right) = E\left(X_0\right)$ as well?

In general, the answer is no as pointed out by [1].The duo envisaged a situation where the player is allowed to go into debt by any amount to play for an arbitrarily long time. In such a situation, the player may inevitably come out ahead. There are conditions which will guarantee fairness and Doyle and Snell give them in the following theorem:

1.2 Theorem [Martingale Stopping Theorem]

A fair game that is stopped at random time will remain fair to the end of the game if it is assumed that;

(a) There is a finite amount of money in the world

(b) A player must stop if he wins all of this money or goes into debt by this amount

A more formal statement of the above theorem with proof is provided in [2] and it is called Doob's Optional-Stopping Theorem. In the exposition, basically, the concepts of [3] and [4] are employed. The statement "there is a finite amount of money in the world" is encoded by assuming that the random variables X_k are uniformly bounded and this is condition (b) below. Similarly, the requirement that the player must stop after a finite amount time is obtained by requiring that τ be almost surely bounded, which is condition (a) below. The third condition under which the theorem holds is essentially limit on the size of a bet at any given time, and this is (c).

1.3 Theorem (Doob's Optional-Stopping Theorem)

Let $\left(\Omega, \mathcal{F}, \left\{\mathcal{F}_k\right\}, P\right)$ be a filtered probability space and $X = \left\{X_k\right\}$ a martingale with respect to $\left\{\mathcal{F}_k\right\}$. Let τ be a stopping time. Suppose that any of the following conditions holds:

(a) There is appositive integer N such that $\tau\left(w\right) \le N \ \forall w \in \Omega$

(b) There is a positive integer K such that $\left|X_k\left(w\right) < K\right| \ \forall k$ and $\forall w \in \Omega$ and τ is a.s. finite

(c) $E\left(\tau\right) < \infty$, and there is a positive real number K such that $\left|X_k\left(w\right) - X_{k-1}\left(w\right)\right| < K \ \forall k$ and $w \in \Omega$.

Then X_τ is integrable and $E\left(X_\tau\right) = E\left(X_0\right)$

II. Stopping Time

Let $\left(\Omega, \mathcal{F}, P\right)$ be a probability space and let $\left\{\mathcal{F}_t\right\}_{t=0}^{n}$ be a filtration. A stopping time is a map (random variable) $\tau : \Omega \mapsto \left\{0,1,2,\ldots,n\right\} \cup \left\{\infty\right\}$ with the property that

$$\left\{\tau = t\right\} = \left\{w \in \Omega : \tau\left(w\right) = t\right\} \in \mathcal{F}_t \ \forall t = 0,1,2,\ldots,n,\infty$$

We say that τ is almost surely finite if $P\left(\left\{\tau = \infty\right\}\right) = 0$.Usually, we set $T = \left\{0,1,2,\ldots,n\right\}$ as the index set for the time variable t, and the σ-algebra $\mathcal{F}_t$ is the collection of events which are observable up to and including time t. The condition that τ is a stopping time means that the outcome of the event $\left\{\tau \le t\right\}$ is known at time t, [8]. No knowledge of the future is required, since such a rule would surely result in an unfair game.

In discrete time situation, where $T = \left\{0,1,2,\ldots,n\right\}$, the condition that $\left\{\tau \le t\right\} \in \mathcal{F}_t$ means that at any time $t \in T$ one knows based on information up to time t if one has been stopped already or not which is equivalent to the requirement that $\left\{\tau = t\right\} \in \mathcal{F}_t$. This is not true for continuous time case in which T is an interval of the real numbers and hence uncountable due to the fact that σ-algebras are not in general closed under taking uncountable unions of events.

2.1 Independent Stopping Time

Let $\{X_t : t \geq 0\}$ be a stochastic process and suppose that t is any random time that is independent of the process. Then t is a stopping time. Here $\{\mathrm{t} = t\}$ does not depend on $\{X_t : t \geq 0\}$ (past or future). An example of this is that before one starts gambling, one decides that one will stop gambling after the 10th gamble regardless of all else. i.e. $P\{\mathrm{t} = 10\}$. Another example is that if every day after looking at the stock price, one throws a fair die. One decides to sell the stock the first time the die shows up an odd number. In this case t is independent of the stock pricing.

2.2 Non-stopping Times (Last Exit Time)

In taking an example of a rat in the open maze problem in which the rat eventually reaches freedom (state 0) and never returns into the maze. Suppose that the rat starts off in cell 1, $X_0 = 1$. Let t denote the last time the rat visits cell 1 before leaving the maze:

$$\mathrm{t} = \max\{t \geq 0 : X_t = 1\} \tag{2.2.1}$$

We now need to know the future to determine such a time. For instance the event $\{\mathrm{t} = 0\}$ tells us that in fact the rat never returned to cell 1

$$\{\mathrm{t} = 0\} = \{X_0 \neq 1, X_1 \neq 1, X_2 \neq 1 ...\} \tag{2.2.2}$$

So, this depends on all of the future, not just X_0. This is not a stopping time. In general, a last exit time (the last time a process hits a given state or set of states) is not a stopping time for in order to know the last has happened or occurred, one must know the future.

Possibly the most popular example of stopping time is the hitting time of a stochastic process, that is that first time at which a certain pre-specified set is hit by the considered process.

The reverse of the above statement is as well true: for any stopping time, there exists an adapted stochastic process and a Borel-measurable set such that the corresponding hitting time will be exactly this stopping time. The statement and other constructions of stopping times are also in [9], [10],[11],[12] and [13].

III. Stochastic Processes

By a stochastic process we mean a collection or family of random variables $X = \{X_t : t \in T\}$ indexed by time, $T \subseteq [0, \infty)$. Since the index t represents time, we then think of X_t as the "state" or the "position" of the process at time t. T, is called the parameter set and $(\square, \square)$, the state of the process. If T is countable, the process is said to be discrete parameter process. If T is not countable, the process is said to be a continuous parameter process.

A random variable t is a stopping time for a stochastic process $\{X_t\}$ if it is a stopping time for the natural filtration of X, that is $\{\mathrm{t} \leq t\} \in \sigma(X_s : s \leq t)$.

The first time that an adapted process X_t, hits a given value or set of values is a stopping time. The inclusion of ∞ into the range of t is to cover all cases where X_t never hits the given values, that is, $\mathrm{t} = \infty$ if the random event never happens. In other words, let $\{X_t, t \geq 0\}$ be a stochastic process. A stopping time with respect to $\{X_t, t \geq 0\}$ is a random time such that for each $t \geq 0$, the event $\{\mathrm{t} = t\}$ is completely determined by (at most) the total information known up the time $t, \{X_0, X_1, ..., X_t\}$.

In the context of gambling, in which X_t denote our total earnings after the gambling at time t, a stopping time t is thus a rule that tells us at what time to stop gambling. Our decision to stop after a given gamble can only depend (at most) on the "information" known at that time (not on the future information). This is to say that a gambler stops the betting process when he reaches a certain goal. The time it takes to reach this goal is generally not a deterministic one. Rather, it is a random variable depending on the current result of the bet, as well as the combined information from all previous bets. If we let t be a stopping time and $X = \{X_t\}$ be random process, then for any positive integer t and $w \in \Omega$, we define

$\tau \wedge t(\omega) = \min\{\tau(\omega), t\}$

3.1 Stopped Process

Given a stochastic process X_k which is adapted to the filtration $\mathcal{F}_k$ and let τ be a stopping time with respect to $\mathcal{F}_k$.

Define the random variable

$$X_{\tau(\omega)} = \begin{cases} X_{\tau(\omega)}, & \tau(\omega) < \infty \\ 0 & \text{otherwise} \end{cases}$$

or equivalently, $X_\tau = \sum_{k=0}^{\infty} X_k 1_{\{\tau = k\}}$.

The process $X_k^\tau = X_{\tau \wedge k}$ is called a stopped process. It is equivalent to X_τ for $\tau \le k$ and equal to X_k for $\tau > k$

Following [14], one can intuitively choose a stochastic process which will assume value 1 until just before the stopping time is reached, from which on it will be 0. The 1-0-process therefore first hits the Borel set {0} at the stopping time. As such a 1-1 relation is established (Theorem 3.5) between stopping times and adapted càdlàg (see 3.2) processes that are non-decreasing and take the values 0 and 1

A 1-0-process as described above is akin to a random traffic light which can go through three possible scenarios over time (with 'green' standing for '1' and 'red' standing for '0') as follows:

(i) Stays red forever (stopped immediately)
(ii) Green at the beginning, then turns red and stays red for the rest of the time (stopped at some stage)
(iii) Stays green forever (never stopped)

For the adaptedness of 1-0-process, it can only change based on information up to the corresponding point in time. This intuitive interpretation of stopping time as the time when such a 'traffic light' lights seems to be easier to understand than the concept of random time which is 'known once it has been reached'.

3.2 Definition

Càdlàg (a French word standing for: continue à droite, limitée à gauche)
Meaning right- continuous with left-handed limit (RCLL).

If for all $\omega \in \Omega$ the paths $X(\omega): T \to \mathbb{R}$ have the property RCLL, a real-valued stochastic process will be called càdlàg.

3.3 Definition

An adapted càdlàg process $X = (X_t)_{t \in T}$ on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a stopping process if

$$X_t(\omega) \in \{0,1\} \qquad (\omega \in \Omega, t \in T) \tag{3.3.1}$$

and

$$X_s \ge X_t \qquad (s \le t, s, t \in T) \tag{3.3.2}$$

For a finite or infinite discrete time axis given by $T = \{t_k : k \in \mathbb{N}, t_k \ge t_j \text{ if } k \ge j\}$, an adapted process satisfying (3.3.1) and (3.3.2) is automatically càdlàg.

3.4 Definition

For a stopping process X on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, we define

$$\tau^X(\omega) = \begin{cases} +\infty & \text{if } X_t(\omega) = 1 \ \forall t \in T \\ \min\{t \in T : X_t(\omega) = 0\} & \text{otherwise} \end{cases} \tag{3.4.1}$$

The minimum in the lower case exists because of the càdlàg property of each path.
By definition

$$X_t = 1_{\{\tau^X > t\}} \qquad (t \in T) \tag{3.4.2}$$

and by adaptedness of X and that $\{\tau^X > t\} = \{\tau^X \le t\}^c$ which implies that

$$\left\{ \mathrm{t}^{X} \le t \right\} \in \mathrm{F}_{\mathrm{t}} \quad \left(t \in T \right) \qquad (3.4.3)$$

Therefore t^{X} is a stopping time for a stopping process X. t^{X} is the first time of X hitting the Lebesgue-measure set {0}.

3.5 Theorem [14].

$$f : X \to \mathrm{t}^{X} \qquad (3.5.1)$$

is a bijection between the stopping processes and the stopping times on $\left(\Omega, \mathrm{F}, \mathrm{F}_{\mathrm{t}}, P\right)$ such that

$$f^{-1} : \mathrm{t} \to X^{\mathrm{t}} \qquad (3.5.2)$$

Hence $\mathrm{t}^{X^{\mathrm{t}}} = \mathrm{t}$ and (3.5.3)

Proof

f maps processes X to stopping times t^{X}. Under f , different stopping processes lead to different stopping times as seen in (3.4.1). Then f is therefore an injection.

For any stopping time □, we have from (3.4.1) and (3.4.2) that

$$\mathrm{t}^{X^{\mathrm{t}}}\left(\mathrm{w}\right) = \begin{cases} +\infty & \text{if } 1_{\{\mathrm{t} > t\}}\left(\mathrm{w}\right) = 1 \quad \forall t \in T \\ \min\left\{ t \in T : 1_{\{\mathrm{t} > t\}}\left(\mathrm{w}\right) = 0 \right\} & \text{otherwise} \end{cases} \qquad (3.5.4)$$

Since $1_{\{\mathrm{t} > t\}}\left(\mathrm{w}\right) = 0 \Leftrightarrow \mathrm{t}\left(\mathrm{w}\right) \le t$, the right hand side of (3.5.4) is $\mathrm{t}\left(\mathrm{w}\right)$. This means that

$\mathrm{t}\left(\mathrm{w}\right) = \mathrm{t}^{X^{\mathrm{t}}}\left(\mathrm{w}\right)$, thus f is surjectve and this proves (3.5.1).

Now, from (3.5.1) and (3.5.3), we have

$$X_t^{\mathrm{t}^{X}} = 1_{\left\{\mathrm{t}^{X} > t\right\}} = X_t \quad \left(t \in T \right)$$

and this proves (3.5.4).

3.6 Theorem

For a stopping time t, the stochastic process $X^{\mathrm{t}} = \left(X_t^{\mathrm{t}} \right)_{t \in T}$ defined by

$$X_t^{\mathrm{t}} = 1_{\{\mathrm{t} > t\}} \qquad \left(t \in T \right) \qquad (3.6.1)$$

is a stopping process.

Proof

By adaptedness of X, we have $\left\{ \mathrm{t}^{X} > t \right\} = \left\{ \mathrm{t}^{X} \le t \right\}^{c}$ and that $\left\{ \mathrm{t} \le t \right\} \in \mathrm{F}_{\mathrm{t}} \quad \left(t \in T \right)$, which implies that X^{t} is an adapted process. We have from the definition that $X_s^{\mathrm{t}} = 1_{\left\{\mathrm{t}^{X} > s\right\}} \ge 1_{\left\{\mathrm{t}^{X} > t\right\}} = X_t^{\mathrm{t}}$ for $s \le t$.

Now,

$\lim_{t \downarrow \mathrm{t}(\mathrm{w})} \left(X_t^{\mathrm{t}}\left(\mathrm{w}\right) \right) = X_{\mathrm{t}(\mathrm{w})}^{t} = 0$ since X^{t} is càdlàg then a stopping process

.

IV. Conclusion

We have in nutshell proved a theorem whose process is càdlàg. One can conclude that many of the processes with such a property can be made to be a stopping process by a stopping time. Thus, it is important to notice that using this construction the result can be extended to continuous processes by perhaps making it càdlàg.

References

[1]. Doyle, P. G. and Snell, J. L. (1984) Random Walks and Electrical Networks. Carus Mathematical Monographs, Mathematical Association of America Washington DC

[2]. Lalonde M. S. (201) The Martingale Stopping Theorem http://math.dartmouth.ed wpw/math100w13

[3]. Williams, D. (1991). Probability with Martingales.Cambridge University Press (Cambridge Mathematical Text Books).

[4]. Doob, J. L. (1971). What is Martingale? American Mathematics Monthly 78 (5) 451-463

[5]. Gut, A. (2004). Probability: A Graduate Course. Springer Texts in Statistics. Springer-verlag.

[6]. Gusak, D., Kukush, A. and Mishura Y, F. (2010). Theory of Stochastic Processes. Springer verlag.

[7]. Strook, D.W. (1993) Probability Theory, an Analytic View.Cambridge University Press.

[8]. Kruglov, V. M. (2011). Stopping Times. Moscow University Computational Mathematics and Cybernetics. **35** (2), 55-62.

[9]. Lutz, P. (2012). Early Stopping – But when? Lecture notes in Computer Science (7700) 53-67.

[10]. Jukka, Lempa (2012). Optimal Stopping with Information Constraint Applied Mathematics and Optimization. **66** (2) 147-173.

[11]. Persi, D. and Laurent, M. (2009). On Times to Quasi-Stationery for Birth and Death Processes. Journal of Theoretical Probability 3 (22) 558-586.

[12]. Neil F. (1980). Construction of Stopping Times T such that $\sigma X_{T} = F_{T} \bmod P$.Lecture Notes in mathematics **794** 412-423

[13]. Nudley, R. M. and Gutman S. (1977). Stopping Times with given Laws.Lecture Notes in mathematics **581** 51-58

[14]. Fischer, T. (2012). On Simple Representations of Stopping Time and Stopping Time σ-algebras. Lecture Notes on Stochastic Processes. Institute of Mathematics, University of Wuerzburg, Germany.