Spectral Continuity: (p, r) - A P And (p, k) - Q

D. Senthil Kumar¹ and P. Maheswari Naik²*

¹Post Graduate and Research Department of Mathematics, Government Arts College (Autonomous), Coimbatore - 641 018, TamilNadu, India.
²Department of Mathematics, Sri Ramakrishna Engineering College, Vattamalaiapalyam, Coimbatore - 641 022, TamilNadu, India.

Abstract: An operator \( T \in B(H) \) is said to be absolute - (p, r) - paranormal operator if
\[
\left\| T^p \left( T^* \right)^r x \right\| \geq \left\| T^r \left( T^* \right)^p x \right\|^p \quad \text{for all } x \in H \text{ and for positive real numbers } p > 0 \text{ and } r > 0 , \text{ where } T = U |T|,
\]
where \( |T| = \sqrt{T^* T} \). In this paper, we prove that continuity of the set theoretic functions spectrum, Weyl spectrum, Browder spectrum and essential surjectivity spectrum on the classes consisting of (p, k) - quasihyponormal operators and absolute - (p, r) - paranormal operators.

Keywords: absolute - (p, r) - paranormal operator, Weyl's theorem, Single valued extension property, Continuity of spectrum, Fredholm, B – Fredholm

I. Introduction and Preliminaries

Let \( H \) be an infinite dimensional complex Hilbert space and \( B(H) \) denote the algebra of all bounded linear operators acting on \( H \). Every operator \( T \) can be decomposed into \( T = U |T| \) with a partial isometry \( U \), where \( |T| = \sqrt{T^* T} \). In this paper, \( T = U |T| \) denotes the polar decomposition satisfying the kernel condition \( N(U) = N(|T|) \). Yamazaki and Yanagida [23] introduced absolute - (p, r) - paranormal operator. It is a further generalization of the classes of both absolute - k - paranormal operators and p - paranormal operators as a parallel concept of class \( A(p, r) \). An operator \( T \in B(H) \) is said to be absolute - (p, r) - paranormal operator, denoted by \( (p, r) - A P \), if
\[
\left\| T^p \left( T^* \right)^r x \right\| \geq \left\| T^r \left( T^* \right)^p x \right\|^p \quad \text{for every unit vector } x \text{ or equivalently}
\]
\[
\left\| T^p \left( T^* \right)^r x \right\| \geq \left\| T^r \left( T^* \right)^p x \right\|^p \quad \text{for all } x \in H \text{ and for positive real numbers } p > 0 \text{ and } r > 0 . \text{ It is also proved that}
\]
\( T = U |T| \) is absolute - (p, r) - paranormal operator for \( p > 0 \) and \( r > 0 \) if and only if \( r \left\| T^r \right\| U \left\| T^p \right\| U \left\| T^r \right\| - (p + r) \lambda^p \left\| T^r \right\|^p + p \lambda^p \left\| T^r \right\|^p \geq 0 \text{ for all real } \lambda \).

a (k, 1) - \( A P \) operator is absolute - k - paranormal;
a (p, p) - \( A P \) operator is p - paranormal;
a (1, 1) - \( A P \) operator is paranormal [23].

An operator \( T \in B(H) \) is said to be (p, k) - quasihyponormal operator, denoted by \( (p, k) - Q \), for some \( 0 < p \leq 1 \) and integer \( k \geq 1 \) if \( T^k (\left\| T \right\|^p - \left\| T^* \right\|^p) T^k \geq 0 \). Evidently,

a (1, k) - \( Q \) operator is k - quasihyponormal;
a (1, 1) - \( Q \) operator is quasihyponormal;
a (p, 1) - \( Q \) operator is k - quasihyponormal or quasi - p - hyponormal ([8], [10]),
a (p, 0) - \( Q \) operator is p - hyponormal if \( 0 < p < 1 \) and hyponormal if \( p = 1 \).

If \( T \in B(H) \), we write \( N(T) \) and \( R(T) \) for null space and range of \( T \), respectively. Let \( \alpha(T) = \dim N(T) = \dim (T^* (0)) \), \( \beta(T) = \dim N(T^*) = \dim (H / T(H)) \), \( \sigma(T) \) denote the spectrum and \( \sigma_a(T) \) denote the approximate point spectrum. Then \( \sigma(T) \) is a compact subset of the set \( C \) of complex numbers. The function \( \sigma \) viewed as a function from \( B(H) \) into the set of all compact subsets of \( C \), with its hausdorff metric, is know to be an upper semi - continuous function [14, Problem 103], but it fails to be continuous [14, Problem 102].
Also we know that $\sigma$ is continuous on the set of normal operators in $B(H)$ extended to hyponormal operators [14, Problem 105]. The continuity of $\sigma$ on the set of quasihyponormal operators (in $B(H)$) has been proved by Erevenko and Djordjevic [10], the continuity of $\sigma$ on the set of $p$ - hyponormal has been proved by Duggal and Djordjevic [9], and the continuity of $\sigma$ on the set of $G_1$ - operators has been proved by Luecke [17].

An operator $T \in B(H)$ is called Fredholm if it has closed range, finite dimensional null space and its range has finite co - dimension. The index of a Fredholm operator is given by $\text{ind}(T) = \alpha(T) - \beta(T)$. The ascent of $T$, $\text{asc}(T)$, is the least non - negative integer $n$ such that $T^n(0) = T^{n+1}(0)$ and the descent of $T$, $\text{dsc}(T)$, is the least non - negative integer $n$ such that $T^n(H) = T^{n+1}(H)$. We say that $T$ is of finite ascent (resp., finite descent) if $\text{asc}(T - \lambda) < \infty$ (resp., $\text{dsc}(T - \lambda) < \infty$) for all complex numbers $\lambda$. An operator $T$ is said to be left semi - Fredholm (resp., right semi - Fredholm), $T \in \Phi_\alpha(H)$ (resp., $T \in \Phi_\beta(H)$) if $\text{TH}$ is closed and the deficiency index $\alpha(T) = \dim(\text{TH}^\perp)$ is finite (resp., the deficiency index $\beta(T) = \dim(\text{H} \cap \text{TH})$ is finite); $T$ is semi - Fredholm if it is either left semi - Fredholm or right semi - Fredholm, and $T$ is Fredholm if it is both left and right semi - Fredholm. The semi - Fredholm index of $T$, $\text{ind}(T)$, is the number $\text{ind}(T) = \alpha(T) - \beta(T)$. An operator $T$ is called Weyl if it is Fredholm of index zero and Browder if it is Fredholm of finite ascent and descent. Let $\mathbb{C}$ denote the set of complex numbers. The Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ of $T$ are the sets $\sigma_w(T) = \{ \lambda \in \mathbb{C} : T - \lambda$ is not Weyl$\}$ and $\sigma_b(T) = \{ \lambda \in \mathbb{C} : T - \lambda$ is not Browder$\}$.

Let $\pi_0(T)$ denote the set of Riesz points of $T$ (i.e., the set of $\lambda \in \mathbb{C}$ such that $T - \lambda$ is Fredholm of finite ascent and descent [7]) and let $\pi_{00}(T)$ and $\pi_{a0}(T)$ denote the set of eigen values of $T$ of finite geometric multiplicity and isolated points of the spectrum. The operator $T \in B(H)$ is said to satisfy Browder's theorem if $\sigma(T) \setminus \sigma_w(T) = \pi_0(T)$ and $T$ is said to satisfy Weyl's theorem if $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$. In [15], Weyl's theorem for $T$ implies Browder's theorem for $T$, and Browder's theorem for $T$ is equivalent to Browder's theorem for $T^*$.

Berkani [5] has called an operator $T \in B(X)$ as $B$ - Fredholm if there exists a natural number $n$ for which the induced operator $T_n : T^n(X) \to T^n(X)$ is Fredholm. We say that the $B$ - Fredholm operator $T$ has stable index if $\text{ind}(T - \lambda \mu) \geq 0$ for every $\lambda, \mu$ in the $B$ - Fredholm region of $T$.

The essential spectrum $\sigma_e(T)$ of $T \in B(H)$ is the set $T \in B(H) = \{ \lambda \in \mathbb{C} : T - \lambda$ is not Fredholm$\}$. Let $\text{acc}(\sigma(T))$ denote the set of all accumulation points of $\sigma(T)$, then $\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_{00}(T) \subseteq \sigma_e(T) \cup \text{acc}(\sigma(T))$. Let $\pi_{a0}(T)$ be the set of $\lambda \in \mathbb{C}$ such that $\lambda$ is an isolated point of $\sigma_a(T)$ and $0 < \alpha(T - \lambda) < \infty$, where $\sigma_a(T)$ denotes the approximate point spectrum of the operator $T$. Then $\pi_0(T) \subseteq \pi_{00}(T) \subseteq \pi_{a0}(T)$. We say that a - Weyl's theorem holds for $T$ if $\sigma_{aw}(T) = \sigma_a(T) \setminus \pi_{a0}(T)$, where $\sigma_{aw}(T)$ denotes the essential approximate point spectrum of $T$ (i.e., $\sigma_{aw}(T) = \bigcap \{ \sigma_a(T + K) : K \in \mathbb{K}(H) \}$ with $\mathbb{K}(H)$ denoting the ideal of compact operators on $H$). Let $\Phi_\alpha(H) = \{ T \in B(H) : \alpha(T) < \infty$ and $T(H)$ is closed$\}$ and $\Phi_\beta(H) = \{ T \in B(H) : \beta(T) < \infty$ \} denote the semigroup of upper semi Fredholm and lower semi Fredholm operators in $B(H)$ and let $\Phi_w(H) = \{ T \in \Phi_\alpha(H) : \text{ind}(T) \leq 0 \}$. Then $\sigma_{aw}(T)$ is the complement in $\mathbb{C}$ of all those $\lambda$ for which $(T - \lambda) \in \Phi_\alpha(H)$ [19]. The concept of a - Weyl's theorem was introduced by Rakocvic [20]. The concept states that a - Weyl's theorem for $T \Rightarrow$ Weyl's theorem for $T$, but the converse is generally false. Let $\sigma_{ab}(T)$ denote the Browder essential approximate point spectrum of $T$.

$$\sigma_{ab}(T) = \bigcap \{ \sigma_a(T + K) : TK = KT \text{ and } K \in \mathbb{K}(H) \} = \{ \lambda \in \mathbb{C} : T - \lambda \notin \Phi_w(H) \text{ or } \text{asc}(T - \lambda) = \infty \}$$
then \( \sigma_{aw}(T) \subseteq \sigma_{ab}(T) \). We say that \( T \) satisfies a - Browder’s theorem if \( \sigma_{ab}(T) = \sigma_{aw}(T) \) [19].

An operator \( T \in B(H) \) has the single valued extension property at \( \lambda_0 \in \mathbb{C} \), if for every open disc \( D \lambda_0 \) centered at \( \lambda_0 \) the only analytic function \( f : D \lambda_0 \to H \) which satisfies

\[
(T - \lambda) f(\lambda) = 0 \quad \text{for all } \lambda \in D \lambda_0,
\]

is the function \( f \equiv 0 \). Trivially, every operator \( T \) has SVEP at points of the resolvent \( \rho(T) = \mathbb{C} \setminus \sigma(T) \); also \( T \) has SVEP at \( \lambda \in \text{iso } \sigma(T) \). We say that \( T \) has SVEP if it has SVEP at every \( \lambda \in \mathbb{C} \). In this paper, we prove that if \( \{T_n\} \) is a sequence of operators in the class \( (p, k) - Q \) or \( (p, r) - A \mathcal{P} \) which converges in the operator norm topology to an operator \( T \) in the same class, then the functions spectrum, Browder spectrum and essential surjectivity spectrum are continuous at \( T \). Note that if an operator \( T \) has finite ascent, then it has SVEP and \( \alpha(T - \lambda) \leq \beta(T - \lambda) \) for all \( \lambda \) \[1, \text{Theorem 3.8 and 3.4}\]. For a subset \( S \) of the set of complex numbers, let \( \overline{S} = \{ \overline{\lambda} : \lambda \in S \} \) where \( \lambda \) denotes the complex number and \( \overline{\lambda} \) denotes the conjugate.

**II. Main Results**

**Lemma 2.1** (i) If \( T \in (p, k) - Q \), then \( \text{asc } (T - \lambda) \leq k \) for all \( \lambda \).

(ii) If \( T \in (p, r) - A \mathcal{P} \), then \( T \) has SVEP.

**Proof:**

(i) Refer [13, Page 146] or [22]

(ii) Refer [21, Theorem 2.8].

**Lemma 2.2** If \( T \in (p, k) - Q \cup (p, r) - A \mathcal{P} \) and \( \lambda \in \text{iso } \sigma(T) \), then \( \lambda \) is a pole of the resolvent of \( T \).

**Proof:** Refer [22, Theorem 6] and [21, Proposition 2.1].

**Lemma 2.3** If \( T \in (p, k) - Q \cup (p, r) - A \mathcal{P} \), then \( T^* \) satisfies a - Weyl’s theorem.

**Proof:** If \( T \in (p, k) - Q \), the \( T \) has SVEP, which implies that \( \sigma(T^*) = \sigma_u(T^*) \) by [1, Corollary 2.45]. Then \( T \) satisfies Weyl’s theorem i.e., \( \sigma(T) \setminus \sigma_w(T) = \pi_0(T) = \pi_{00}(T) \) by [13, Corollary 3.7]. Since \( \pi_{00}(T) = \pi_{00}(T^*) = \pi_{00}(T^*) \), \( \sigma(T) = \sigma(T^*) = \sigma_u(T^*) \) and \( \sigma_w(T) = \sigma_u(T^*) = \sigma_{au}(T^*) \) by [3, Theorem 3.6(ii)], \( \sigma_u(T^*) \setminus \sigma_{au}(T^*) = \pi_{00}(T^*) \). Hence if \( T \in (p, k) - Q \), then \( T^* \) satisfies a - Weyl’s theorem.

If \( T \in (p, r) - A \mathcal{P} \), then by [21, Theorem 2.18], \( T^* \) satisfies a - Weyl’s theorem.

**Corollary 2.4** If \( T \in (p, k) - Q \cup (p, r) - A \mathcal{P} \) and \( \lambda \in \sigma_u(T^*) \setminus \sigma_{au}(T^*) \Rightarrow \lambda \in \text{iso } \sigma_u(T^*) \).

**Lemma 2.5** If \( T \in (p, k) - Q \cup (p, r) - A \mathcal{P} \), then \( \text{asc } (T - \lambda) < \infty \) for all \( \lambda \).

**Proof:** Since \( T - \lambda \) is lower semi - Fredholm, it has SVEP. We know that from [1, Theorem 3.16] that SVEP implies finite ascent. Hence the proof.

**Lemma 2.6** [6, Proposition 3.1] If \( \sigma \) is continuous at a \( T^* \in B(H) \), then \( \sigma \) is continuous at \( T \).

**Lemma 2.7** [12, Theorem 2.2] If an operator \( T \in B(H) \) has SVEP at points \( \lambda \in \sigma_u(T) \), then \( \sigma \) is continuous at \( T \leftrightarrow \sigma_{au} \) is continuous at \( T \).

**Lemma 2.7** If \( \{T_n\} \) is a sequence in \( (p, k) - Q \) or \( (p, r) - A \mathcal{P} \) which converges in norm to \( T \), then \( T^* \) is a point of continuity of \( \sigma_{au} \).

**Proof:** We have to prove that the function \( \sigma_{au} \) is both upper semi - continuous and lower semi - continuous at \( T^* \). But by [11, Theorem 2.1], we have that the function \( \sigma_{au} \) is upper semi - continuous at \( T^* \). So we have to
prove that \( \sigma_{ea} \) is lower semi-continuous at \( T^* \) i.e., \( \sigma_{ea}(T^*) \subseteq \lim \inf \sigma_{ea}(T_n^*) \). Assume the contradiction that \( \sigma_{ea} \) is not lower semi-continuous at \( T^* \). Then there exists an \( \epsilon > 0 \), an integer \( n_0 \), a \( \lambda \in \sigma_{ea}(T^*) \) and an \( \epsilon \)-neighbourhood \( (\lambda, \epsilon) \_T \) such that \( \sigma_{ea}(T_n^*) \cap (\lambda, \epsilon) \_T = \emptyset \) for all \( n \geq n_0 \). Since \( \lambda \notin \sigma_{ea}(T_n^*) \) for all \( n \geq n_0 \) implies \( T_n^* - \lambda \in \Phi_{ea}(H) \) for all \( n \geq n_0 \), the following implications holds:

\[
\text{ind}(T_n^* - \lambda) \leq 0, \quad \alpha (T_n^* - \lambda) < \infty, \quad \text{and} \quad (T_n^* - \lambda) H \text{ is closed} \\
\Rightarrow \text{ind}(T_n - \lambda) \geq 0, \quad \beta (T_n - \lambda) < \infty \\
\Rightarrow \text{ind}(T_n - \lambda) = 0, \quad \alpha (T_n - \lambda) < \beta (T_n - \lambda) < \infty
\]

(Since \( T_n \in (p, k) - Q \cup (p, r) - A \mathcal{P} \Rightarrow \text{ind}(T_n - \lambda) \leq 0 \) by Lemma 2.1 and Lemma 2.5.

for all \( n \geq n_0 \). The continuity of the index implies that \( \text{ind}(T - \lambda) = \lim_{n \to \infty} \text{ind}(T_n - \lambda) = 0 \), and hence that \( (T - \lambda) \) is Fredholm with \( (T - \lambda) = 0 \). But then \( T^* - \lambda \) is Fredholm with \( \text{ind}(T^* - \lambda) = 0 \Rightarrow T^* - \lambda \in \Phi_{ea}(H) \), which is a contradiction. Therefore \( \sigma_{ea} \) is lower semi-continuous at \( T^* \). Hence the proof.

**Theorem 2.9** If \( \{T_n\} \) is a sequence in \( (p, k) - Q \) or \( (p, r) - A \mathcal{P} \) which converges in norm to \( T \), then \( \sigma \) is continuous at \( T \).

**Proof:** Since \( T \) has SVEP by Lemma 2.1, \( \sigma(T^*) = \sigma_a(T^*) \). Evidently, it is enough if we prove that \( \sigma_a(T_n^*) \subseteq \lim \inf \sigma_a(T_n^*) \) for every sequence \( \{T_n\} \) of operators in \( (p, k) - Q \) or \( (p, r) - A \mathcal{P} \) such that \( T_n \) converges in norm to \( T \). Let \( \lambda \in \sigma_a(T^*) \). Then either \( \lambda \in \sigma_{ea}(T^*) \) or \( \lambda \in \sigma_a(T^*) \setminus \sigma_{ea}(T^*) \).

If \( \lambda \in \sigma_{ea}(T^*) \), then proof follows, since

\[
\sigma_{ea}(T^*) \subseteq \lim \inf \sigma_{ea}(T_n^*) \subseteq \lim \inf \sigma_a(T_n^*)
\]

If \( \lambda \in \sigma_a(T^*) \setminus \sigma_{ea}(T^*) \), then \( \lambda \in \sigma_a(T^*) \) by Corollary 2.4. Consequently, \( \lambda \in \lim \inf \sigma_a(T_n^*) \) i.e., \( \lambda \in \lim \inf \sigma_a(T_n^*) \) for all \( n \) by [16, Theorem IV. 3.16], and there exists a sequence \( \{\lambda_n\} \), \( \lambda_n \in \sigma_a(T_n^*) \), such that \( \lambda_n \) converges to \( \lambda \). Evidently \( \lambda \in \lim \inf \sigma_a(T_n^*) \). Hence \( \lambda \in \lim \inf \sigma_a(T_n^*) \). Now by applying Lemma 2.6, we obtain the result.

**Corollary 2.10** If \( \{T_n\} \) is a sequence in \( (p, k) - Q \) or \( (p, r) - A \mathcal{P} \) which converges in norm to \( T \), then \( \sigma \), \( \sigma_a \), and \( \sigma_b \) are continuous at \( T \).

**Proof:** Combining Theorem 2.9 with Lemma 2.7 and Lemma 2.8, we obtain the results.

Let \( \sigma_e(T) = \{ \lambda : T - \lambda \text{ is not surjective} \} \) denote the surjectivity spectrum of \( T \) and let \( \Phi_{ea}(H) = \{ \lambda : T - \lambda \in \Phi_e(H), \text{ind}(T - \lambda) \geq 0 \} \). Then the essential surjectivity spectrum of \( T \) is the set \( \sigma_{ea}(T) = \{ \lambda : T - \lambda \notin \Phi_{ea}(H) \} \).

**Corollary 2.11** If \( \{T_n\} \) is a sequence in \( (p, k) - Q \) or \( (p, r) - A \mathcal{P} \) which converges in norm to \( T \), then \( \sigma_{ea}(T) \) is continuous at \( T \).

**Proof:** Since \( T \) has SVEP by Lemma 2.1, \( \sigma_{ea}(T) = \sigma_{ea}(T^*) \) by [1, Theorem 3.65 (ii)]. Then by applying Lemma 2.8, we obtain the result.

Let \( K \subseteq B(H) \) denote the ideal of compact operators, \( B(H) \) the Calkin algebra and let \( \pi : B(H) \to B(H) / K \) denote the quotient map. Then \( B(H) / K \) being a \( C^* \)-algebra, there exists a Hilbert space \( H_1 \) and an
isometric * - isomorphism $\nu : B(H) / K \to B(H)$ such that the essential spectrum $\sigma_e(T) = \sigma(\pi(T))$ of $T \in B(H)$ is the spectrum of $\nu \circ \pi(T)$ ( $\in B(H)$). In general, $\sigma_e(T)$ is not a continuous function of $T$.

**Corollary 2.12** If $\{\pi(T_n)\}$ is a sequence in $(p, k) - Q$ or $(p, r) - A P$ which converges in norm to $\pi(T)$, then $\sigma_e(T)$ is continuous at $T$.

**Proof:** If $T_n \in B(H)$ is essentially $(p, k) - Q$ or $(p, r) - A P$, i.e., if $\pi(T_n) \in (p, k) - Q$ or $(p, r) - A P$, and the sequence $\{T_n\}$ converges in norm to $T$, then $\nu \circ \pi(T) (\in B(H))$ is a point of continuity of $\sigma$ by Theorem 2.9. Hence $\sigma_e$ is continuous at $T$, since $\sigma_e(T) = \sigma(\nu \circ \pi(T))$.

Let $H(\sigma(T))$ denote the set of functions $f$ that are non-constant and analytic on a neighbourhood of $\sigma(T)$.

**Lemma 2.13** Let $T \in B(X)$ be an invertible $(p, r) - A P$ and let $f \in H(\sigma(T))$. Then $\sigma_{bw}(f(T)) \subset f(\sigma_{bw}(T))$, and if the $B$ - Fredholm operator $T$ has stable index, then $\sigma_{bw}(f(T)) = f(\sigma_{bw}(T))$.

**Proof:** Let $T \in B(X)$ be an invertible $(p, r) - A P$, let $f \in H(\sigma(T))$, and let $g(T)$ be an invertible function such that $f(\mu) - \lambda = (\mu - \alpha_1)......(\mu - \alpha_n)g(\mu)$. If $\lambda \notin f(\sigma_{bw}(T))$, then $f(T) - \lambda = (T - \alpha_1)......(T - \alpha_n)g(T)$ and $\alpha_i \notin \sigma_{bw}(T)$, $i = 1, 2, ..., n$. Consequently, $T - \alpha_i$ is a B - Fredholm operator of zero index for all $i = 1, 2, ..., n$, which, by [5, Theorem 3.2], implies that $f(T) - \lambda$ is a B - Fredholm operator of zero index. Hence, $\lambda \notin \sigma_{bw}(f(T))$.

Suppose now that $T$ has stable index, and that $\lambda \notin \sigma_{bw}(f(T))$. Then, $f(T) - \lambda = (T - \alpha_1)......(T - \alpha_n)g(T)$ is a B - Fredholm operator of zero index. Hence, by [4, Corollary 3.3], the operator $g(T)$ and $T - \alpha_i$, $i = 1, 2, ..., n$, are B - Fredholm and

$$0 = \text{ind} (f(T) - \lambda) = \text{ind} (T - \alpha_1) + .... + \text{ind} (T - \alpha_n) + \text{ind} g(T).$$

Since $g(T)$ is an invertible operator, $\text{ind} g(T) = 0$; also $\text{ind} (T - \alpha_i)$ has the same sign for all $i = 1, 2, ..., n$. Thus $\text{ind} (T - \alpha_i) = 0$, which implies that $\alpha_i \notin \sigma_{bw}(T)$ for all $i = 1, 2, ..., n$, and hence $\lambda \notin f(\sigma_{bw}(T))$.

**Lemma 2.14** Let $T \in B(X)$ be an invertible $(p, r) - A P$ has SVEP, then $\text{ind} (T - \lambda) \leq 0$ for every $\lambda \in C$ such that $T - \lambda$ is B - Fredholm.

**Proof:** Since $T$ has SVEP by [21, Theorem 2.8], then $T|_{B(X)}$ has SVEP for every invariant subspaces $M \subset X$ of $T$. From [4, Theorem 2.7], we know that if $T - \lambda$ is a B - Fredholm operator, there exist $T - \lambda$ invariant closed subspaces $M$ and $N$ of $X$ such that $X = M \oplus N$, $(T - \lambda)|_{M}$ is a Fredholm operator with SVEP and $(T - \lambda)|_{N}$ is a Nilpotent operator. Since $\text{ind} (T - \lambda)|_{M} \leq 0$ by [18, Proposition 2.2], it follows that $\text{ind} (T - \lambda) \leq 0$.

References


DOI: 10.9790/5728-11111318 www.iosrjournals.org 17 | Page
Spectral continuity: \((p, r) - A\) and \((p, k) - Q\)


