IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 10, Issue 5 Ver. VI (Sep-Oct. 2014), PP 05-09
www.iosrjournals.org

On Graph of a Finite Group

R.A.Muneshwar[1], K.L.Bondar[2]
[1]*Department of Mathematics, NES Science College, SRTMU Nanded, India – 431602*
[2]*Department of Mathematics, NES Science College, SRTMU Nanded, India – 431602*

Abstract: *In this paper we introduced a new concept of graph of any finite group and we obtained graphs of some finite groups. Moreover some results on this concept are proved.*

Keywords: *Group, Abelian group, Cyclic group, Graph, Degree of a graph.*

I. Introduction:

The origin of graph theory started with the problem of Koinsberg bridge, in 1735. This problem lead to the concept of Eulerian graph. Euler studied the problem of Koinsberg bridge and constructed a structure to solve the problem called Eulerian graph. In 1840, A.F Mobius gave the idea of complete graph and bipartite graph and Kuratowski proved that they are planar by means of recreational problems. The concept of tree was implemented by Gustav Kirchhoff in 1845, and he employed graph theoretical ideas in the calculation of currents in electrical networks or circuits. In 1852, Thomas Gutherie found the famous four color problem. Then in 1856, Thomas. P. Kirkman and William R.Hamilton studied cycles on polyhydra and invented the concept called Hamiltonian graph by studying trips that visited certain sites exactly once. In 1913, H.Dudeney mentioned a puzzle problem. Eventhough the four color problem was invented it was solved only after a century by Kenneth Appel and Wolfgang Haken. This time is considered as the birth of Graph Theory [1].

Caley studied particular analytical forms from differential calculus to study the trees. This had many implications in theoretical chemistry. This lead to the invention of enumerative graph theory. Any how the term "Graph" was introduced by Sylvester in 1878 where he drew an analogy between "Quantic invariants" and covariants of algebra and molecular diagrams. In 1941, Ramsey worked on colorations which lead to the identification of another branch of graph theory called extremel graph theory. In 1969, the four color problem was solved using computers by Heinrich. The study of asymptotic graph connectivity gave rise to random graph theory.

Graph theory is rapidly moving into the mainstream of mathematics mainly because of its applications in diverse fields which include biochemistry, electrical, engineering (communications networks and coding theory), computer science (algorithms and computations) and operations research (scheduling). The powerful combinatorial methods found in graph theory have also been used to prove significant and well-known results in a variety of areas in mathematics itself. An application of matching in graph theory shows that there is a common set of left and right coset representatives of a subgroup in a finite group [1].

Up to this point, we have been looking at a group as a collection of elements that satisfy some conditions. Because graph has wide range of application in various fields, this motivates us to convert group into graph and make it applicable into various field. In this paper we try to bring very different way of representing the group, using the graph associated with the group rather than the algebraic structure of group. This paper is meant as an introduction and overview of some nice ideas from group theory by using graph theory. We convert group into graph and try to study various properties of group by using graph theory [1].

II. Some Basic Definitions:

Following definitions are comes from references [2], [3], [4], [5], [6], [7], [8].

Definition 2.1 (Group): A nonempty set G with a binary operation . is called as a group if the following axioms hold:

(i) $a(bc) = (ab)c$ for all $a,b,c \in G$
(ii) There exists e in G such that $ea = ae = a$; $\forall a \in G$
(iii) For every $a \in G$ there exists $a' \in G$ such that $a' a = a a' = e$.

Definition 2.2 (Abelian group): A group G in which all elements satisfies commutative law is called as a abelian group.

Definition 2.3 (Cyclic group): A group G is said to be cyclic if $G = [a] = \{x=a^n \mid n \in Z\}$, for some $a \in G$.The most important examples of cyclic groups are the additive group Z of integers and the additive groups Z/(n) of integers modulo n. In fact, these are the only cyclic groups up to isomorphism.

Definition 2.4 (Subgroup): Let (G, .) be a group and H be a subset of G. Then H is called a subgroup of G, if H is a group relative to the binary operation in G and it is denoted by $H \leq G$.
Definition 2.5 (Center of a group): The center of a group G, written as Z(G), is the set of those elements in G that commute with every clement in G. That is $Z(G) = (a \in G \mid ax = xa \ \forall x \in G \}$.
Definition 2.6 (Centralizer of an element): Let $g \in G$ be any elements of group G then centralizer of an element is written as C(g), is the set of those element in G that commute with element g . i.e.$C(g)=\{a \in G \mid ag = ga , g \in G \}$.
Definition 2.7 (Centralizer of a subgroup): Let H be any subgroup of G then centralizer of a subgroup is written as C(H), is the set of those elements in G that commute with all elements of subgroup H.
i.e. $C(H) = \{a \in G \mid ah = ha , \ \forall h \in H \}$.
Definition 2.8 (Order of a element): Let G be a group, and $a \in G$. If there exists a least positive integer m such that $a^m = e$, then such positive integer m is called as order of a and it is written as o(a). If no such positive integer exists, then a is said to be of infinite order.
Definition 2.9 (Order of a group): Number of elements in a group G is called as order of a group and it is denoted by o(G) or |G|. If order of a group is finite then group is said to be finite group and if order of a group is infinite then group is said to be infinite group.

Definition 2.10 (Graph): Graph be an ordered pair G = (V, E), where V be a set of vertices of graph and E be a set of edges of graph. The vertices $g_{i,}$,g_j associated with edge e_k are called as end vertices of e_k.
Definition 2.11 (Degree of a vertex): Number of edges incident on vertex g_k with loop counted twice is called as degree of a vertex g_k, and it is denoted by $d(g_k)$.
Definition 2.12 (Degree of a graph): Sum of degree of all vertices of a graph is called as degree of a graph and it is denoted by d(G).
Definition 2.13 (Adjacent vertices): If g_i and g_j are end vertices of some edge e then such vertices are called as adjacent vertices.

Definition 2.14 (Loop): An edge having the same end vertices is called as loop.
Definition 2.15 (Parallel edges): A pair of edges with same end vertices is called as parallel edges.
Definition 2.16 (Simple graph): A graph that has neither self loop nor parallel edges is called as simple graph.
Definition 2.17 (Regular graph): A graph in which degree of every vertex is same then is called as regular graph.
Definition 2.18 (Complete graph): A graph in which every vertex of graph G is adjacent to all other vertices of graph is called as complete graph.
Note: Every complete graph is regular graph but converse is not true.

III. Graph of a finite group:

Let G is a finite group of order n. Then graph of G is denoted by R(G) and is defined as R(G) = (R(V), R(E)), where
1. R(V) = set of vertex of graph of G = G. and
2. R(E) = set of edges of graph of G
= $\{r_{ij} \mid r_{ij}$ is an edge between g_i and g_j if and only if g_i and g_j are commutes in group $\}$
Example 3.1: If $G = Z_4 = \{ 0,1,2,3 \}$ be an abelian group of order 4. Then by let R(G) = { R(V), R(E) } be graph of a group G, where $R(V) = G = Z_4 = \{ 0,1,2,3 \}$ and
$R(E) = \{ r_{ij} = (g_{i,} g_j) \mid g_i$ and g_j are commute in a group $\}$
i.e. $R(E) = \{ r_{ij} = (g_{i,} g_j) \mid g_i g_j = g_j g_i \ \forall \ i, j \}$
Hence R(E) = { (0,0), (1,1), (2,2), (3,3), (0,1), (0,2), (0,3), (1,2), (1,3),(2,3)}
Thus the graph of G is as follows.

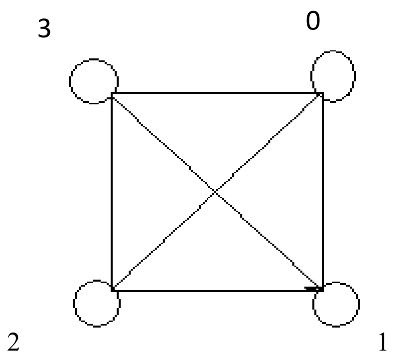

Fig. 1

Example 3.2: If $G = D_3 = \{e, a, a^2, r_1, r_2, r_3\}$ be a non abelian group of order 6.
Let $R(G) = \{ R(V), R(E) \}$ be graph of a group G, where $R(G) = G = D_3 = \{e, a, a^2, r_1, r_2, r_3\}$ and
$R(E) = \{ r_{ij} = (g_i, g_j) \mid g_i$ and g_j are commute in a group $G \}$
$R(E) = \{ r_{ij} = (g_i, g_j) \mid g_i g_j = g_j g_i \; \forall \; i, j \}$
Hence $R(E) = \{ (e, e), (e, a), (e, a^2), (e, r_1), (e, r_2), (e, r_3), (a, a), (a, a^2), (r_1,r_1), (r_2,r_2), (r_3,r_3) \}$
Thus the graph representation of $G = D_3$ is as follows:

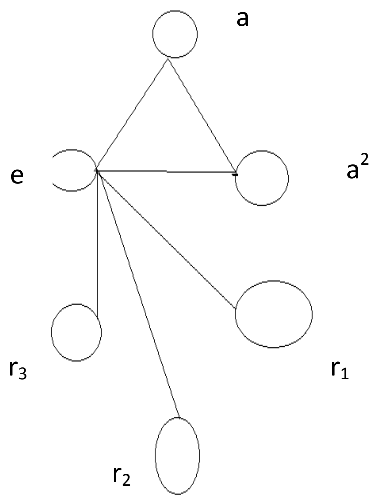

Fig. 2

IV. Some Results on graph of a finite group:

Theorem 4.1 If G be any group of order n then $o(C(g_i)) = d(g_i) -1$.
Proof: Let G be any group of order n, then the graph of a group is an ordered pair $R(G)= (R(V), R(E))$. Then by definition of $C(g_i)$, we have

$C(g_i) = \{ g_k \in G \mid g_i g_k = g_k g_i$ for a fix $i \}$
$= \{ g_k \in G \mid$ there is an edge between vertices g_i and g_k in graph a fix $i \}$
$= \{ g_k \in G \mid (g_i , g_k)$ in $R(E) \}$
$= \{ g_k \in G \mid g_k$ is one end of edge (g_i , g_k) for fix $i \}$
$= \{ g_k \in G \mid g_k$ is adjacent to g_i for fix $i \}$

Hence $o(C(g_i)$ = Number of those vertices which are adjacent to g_i .
i.e. $o(C(g_i)$= Number of those edges whose one (or both) end is g_i .
But $d(g_i)$ = Number of those edges which are incident on g_i with loop counted twice.
i.e. $d(g_i)$ = Number of those edges whose one end is g_i with loop counted twice.
i.e. $d(g_i)$ = Number of those vertices which are adjacent to g_i .

Hence $o(C(g_i))$ = order of centralizer of an $g_i = d(g_i) - 1$.
i.e. $d(g_i) = o(C(g_i)) + 1$.

Now we can define centralizer of an element in the form of graph of a group G. Let $g_i \in G$ be any vertex of a graph of group G, then centralizer of vertex is denoted by $C(g_i)$ and defined as,

$C(g_i) = \{ g_k \in G \mid g_i g_k = g_k g_i \text{ for } i \}$
$= \{ g_k \in G \mid$ there is an edge between vertices g_i and g_k in graph for a fix i $\}$
$= \{ g_k \in G \mid (g_i , g_k) \in E(G) \}$
$= \{ g_k \in G \mid g_k$ is one end of edge (g_i , g_k) for a fix i $\}$
$= \{ g_k \in G \mid g_k$ is adjacent to g_i for fix i $\}$
= Collection of those vertices of graph of group G which are adjacent to g_i

Theorem 4.2. If G be any group of order n with identity elements e then $d(e) = o(G) + 1$.
Proof : Let G be any group of order n with identity elements e. Then graph of group is an ordered pair R(G)= (R(V), R(E)),
Then by definition of $C(g_i)$, we have
$C(e) = \{ g_k \mid e\, g_k = g_k\, e \}$
$= G$.
Hence $o(C(e)) = o(G)$.
By Theorem 4.1, $d(e) = o(C(e)) + 1 = o(G)+1$.
Thus $o(C(e)) = d(e) - 1$.
i.e. $o(G) = d(e) - 1$.

Theorem 4.3 If G be any group of order n then $d(G) = \sum_{k=1}^{n} o(C(gi)) + n$.
Proof : Let G be any group of order n, then the graph of group G is an ordered pair R(G)= (R(V), R(E)),
d(G) = Degree of graph G
= Sum of degree of all vertices of graph of G
$= \sum_{k=1}^{n} d(g_k) = \sum_{k=1}^{n} (o(C(gi)) + 1)$
$= \sum_{k=1}^{n} o(C(gi)) + n$.
Hence $\sum_{k=1}^{n} o(C(gi)) = d(G) - n$.

Theorem 4.4: If G be any abelian group of order n then $d(g_i) = o(G) + 1 \quad ; \forall g_i \in G$
Proof : Let G be any abelian group of order n, then we have
$g_i g_k = g_k g_i \quad ; \forall g_k, g_i \in G$
hence $C(g_i) = \{ g_k \mid g_i g_k = g_k g_i \text{ for fix } i \}$
$= G$
Hence $o(C(g_i)) = o(G) \,; \forall g_i \in G$
By Theorem 4.1, we have $d(g_i) = o(C(g_i)) + 1 = o(G) + 1 \,; \forall g_i \in G$
Thus $o(G) = d(g_i) - 1 \,; \forall g_i \in G$

Theorem 4.5: If G be any abelian group of order n then $d(G) = o(G)(o(G)+ 1)$
Proof : Let G be any abelian group of order n, we have
$g_i g_k = g_k g_i \quad ; \forall g_k, g_i \in G$
Let the graph of group G is an ordered pair R(G)= (R(V), R(E)),
By definition of degree of a graph we have
$d(G)$ = Sum of $d(g_i)$ for all g_i in G
$= d(g_1) + d(g_2) + \ldots\ldots\ldots + d(g_n)$
By Theorem 4.1 we have $d(g_i) = o(G) + 1 \quad \forall g_i \in G$
hence, $d(G) = (o(G) + 1) + (o(G) + 1) + \ldots\ldots\ldots + (o(G) + 1)$ (n times)
$= o(G) [(o(G) + 1)]$
Since every cyclic group is abelian group, we have fallowing corollary.
Corollary 4.6 : If G be a Cyclic group of order n then
1. $d(g_i) = o(C(g_i)) + 1. = o(G) + 1 \,; \forall g_i \in G$
2. $d(G) = o(G)(o(G)+ 1)$

V. Conclusion:

An attempt has been made to show that graphs can be used to represent almost any problem involving discrete arrangements of objects, where concern is not with the internal properties of these objects but with the relationships among them. We try to made relationship between graph theory and group theory. Moreover we try to represent group as graph and study various properties of group by using corresponding graph of group.

References:

[1]. S.G. Shrinivas et. al. / International Journal of Engineering Science and Technology, Vol. 2(9), 2010, 4610-4621 /

[2]. Joseph A. Gillian, Contemporary Abstract Algebra (Fourth Edition, Narosa Publishing House Pvt. Ltd. 1999)

[3]. I. S. Luther and I. B. S. Passi, Algebra-Vol. 1: Groups (Narosa Publishing House Pvt. Ltd., New Delhi, 1996.)

[4]. Jean H. Gallier. Discrete Mathematics.(Universitext. Springer Verlag, first edition, 2011.)

[5]. Reinhard Diestel, Graph Theory (Springer International Publication, Second Edition, 2011)

[6]. Douglas B. West, Introduction to graph theory (Eastern Economy Edition, PHL Learning Private Limited, Second Edition, 2011)

[7]. Narsingh Deo, Graph Theory with applications to engineering and computer science (Eastern Economy Edition, Prentice - Hall Of India Private Limited, 2005)

[8]. Pratima Panigrahi, S.B. Rao, Graph Theory Research Directions (Narosa Publishing House Pvt. Ltd. 2011).