On Jordan Generalized Higher Reverse Derivations on \( \Gamma \)-rings

Salah Mehdi Salih and Marwa Riyadh Salih

Al-Mustansiryia University College of Education Department of Mathematics

Abstract: In this paper, we study the concepts of generalized higher reverse derivation and Jordan generalized higher reverse derivation and Jordan generalized triple higher reverse derivation on \( \Gamma \)-ring \( M \).

The aim of this paper is to prove that every Jordan generalized higher reverse derivation of \( \Gamma \)-ring \( M \) is generalized higher reverse derivation of \( M \).

Mathematics Subject Classification: 16U80, 16W25

Key word: \( \Gamma \)-ring, prime \( \Gamma \)-ring, semiprime \( \Gamma \)-ring, derivation, higher derivation, generalized higher derivation of \( \Gamma \)-ring, reverse derivation of \( R \)

I. Introduction

The concepts of a \( \Gamma \)-ring was first introduced by Nobusae[9] in 1964 this \( \Gamma \)-ring is generalized by W.E.Barnesin [2] a broad sense that served now a day to call a \( \Gamma \)-ring.

Let \( M \) and \( \Gamma \) be two additive abelian groups. Suppose that there is a mapping from \( M \times \Gamma \times M \rightarrow M \) (the image of \((a,\alpha,\beta)\) being denoted by \( a\alpha \beta \, \alpha,\beta \in M \) and \( \alpha,\beta \in \Gamma \)) satisfying for all \( a,b,c \in M \) and \( \alpha,\beta \in \Gamma \)

\[
i(a+b)\alpha c = a\alpha c + b\alpha c
\]

\[
a(\alpha + \beta) c = a\alpha c + a\beta c
\]

\[
a\alpha(b + c) = a\alpha b + a\alpha c
\]

Then \( M \) is called a \( \Gamma \)-ring.[2]

Throughout this paper \( M \) denotes a \( \Gamma \)-ring with center \( Z(M) \) [1], recall that a -\( \Gamma \)-ring \( M \) is called prime if \( a\Gamma \Gamma M =0 \) implies \( a=0 \) or \( b=0 \) [8], and it is called semiprime if \( a\Gamma \Gamma M =0 \) implies \( a=0 \) [6], a prim \( \Gamma \)-ring is obviously semiprime and a \( \Gamma \)-ring \( M \) is called 2-torsion free if \( 2a=0 \) implies \( a=0 \) for every \( a \in M \) [5], an additive mapping \( d \) from \( M \) into itself is called a derivations if \( d(ab) = d(a)b + ad(b), \) for all \( a,b \in M, \alpha \in \Gamma \) [7] and \( d \) is said to be Jordan derivation of a \( \Gamma \)-ring \( M \) if \( d(a\alpha a) = d(a)a\alpha a + a\alpha d(a), \) for all \( a \in M, \alpha \in \Gamma \) [7]. A mapping \( f \) from \( M \) into itself is called generalized derivation of \( M \) if there exists derivation \( d \) of \( M \) such that \( f(ab) = f(a)ab + a\alpha d(b), \) for all \( a,b \in M, \alpha \in \Gamma \). And \( f \) is said to be Jordan generalized derivation of \( \Gamma \)-ring \( M \) if there exists derivation \( d \) of \( M \) such that \( f(a\alpha a) = f(a)a\alpha a + a\alpha d(a) \) for all \( a \in M \) and \( \alpha \in \Gamma \).

Bresar and Vukman[3] have introduced the notion of a reverse derivation as an additive mapping \( d \) from a ring \( R \) into itself satisfying \( d(xy) = d(y)x + yd(x) \) for all \( x,y \in R \).

M. Sammn[10] presented the study between the derivation and reverse derivation in semiprime ring \( R \). Also it is shown that non-commutative prime rings don't admit a non-trivial skew commuting derivation.

We defined in [11] the concepts of higher reverse derivation of \( \Gamma \)-ring \( M \) as follow:

Let \( D=(d_n)_{n \in N} \) be additive mappings on a ring \( R \) then \( D \) is called higher reverse derivation of \( \Gamma \)-ring \( M \) if

\[
d_n(x\alpha y) = \sum_{i+j=n} d_i(y)\alpha d_j(x)
\]

For all \( x,y \in M, \alpha \in \Gamma \) and \( n \in N \)

and Jordan higher reverse derivation of \( \Gamma \)-ring \( M \) if

\[
d_n(x\alpha x) = \sum_{i+j=n} d_i(x)\alpha d_j(x)
\]

and Jordan triple higher reverse derivation of \( \Gamma \)-ring \( M \) if

\[
d_n(x\alpha y\beta x) = d_n(x)\beta x\alpha y + \sum_{i+j+r=n} d_i(x)\beta d_j(y)\alpha d_r(x)
\]

For all \( x,y \in M, \alpha, \beta \in \Gamma \) and \( n \in N \)

also we proved that every Jordan higher reverse derivation of a \( \Gamma \)-ring \( M \) is higher reverse derivation of \( M \) [11], the main object of this paper is present the concepts of generalized higher reverse derivation, Jordan
generalized higher reverse derivation of \( \Gamma \)-ring \( M \) and we prove that every Jordan generalized higher reverse derivation of \( \Gamma \)-ring \( M \) is generalized higher reverse derivation of \( M \).

II. Generalized Higher Reverse Derivation of \( \Gamma \)-Rings

In this section we introduce and study of concepts of generalized higher reverse derivation, Jordan generalized higher reverse derivation and Jordan generalized triple higher reverse derivation of \( \Gamma \)-ring.

Definition 2.1:
Let \( M \) be a \( \Gamma \)-ring and \( F = (f_i)_{i \in \mathbb{N}} \) be a family of additive mappings of \( M \) such that \( f_0 = \text{id}_M \) then \( F \) is called \textbf{generalized higher reverse derivation of} \( M \) if there exists a higher reverse derivation \( D = (d_i)_{i \in \mathbb{N}} \) of \( M \) such that for all \( n \in \mathbb{N} \) we have:
\[
f_n(x \alpha y) = \sum_{i+j=n} f_i(y) \alpha d_j(x) \ldots \ldots (i)
\]
\( F \) is called \textbf{a Jordan generalized higher reverse derivation of} \( M \) if there exists a Jordan higher reverse derivation \( D = (d_i)_{i \in \mathbb{N}} \) of \( M \) such that for all \( n \in \mathbb{N} \) we have:
\[
f_n(x \alpha x) = \sum_{i+j=n} f_i(x) \alpha d_j(x) \ldots \ldots (ii)
\]
For every \( x,y \in M \) and \( \alpha \in \Gamma \)
\( F \) is said to be \textbf{a Jordan generalized triple higher reverse derivation of} \( M \) if there exists Jordan triple higher reverse derivation \( D = (d_i)_{i \in \mathbb{N}} \) of \( M \) for all \( n \in \mathbb{N} \) we have:
\[
f_n(x \alpha y \beta x) = f_n(x) \beta x \alpha y + \sum_{i+j+r=n} f_i(x) \beta d_j(y) \alpha d_r(x) \ldots \ldots (iii)
\]
For every \( x,y \in M \) and \( \alpha, \beta \in \Gamma \)

Example 2.2:
Let \( F = (f_i)_{i \in \mathbb{N}} \) be a generalized higher reverse derivation on a ring \( R \) then there exists a higher reverse derivation \( d = (f_i)_{i \in \mathbb{N}} \) of \( R \) such that
\[
f_n(xy) = \sum_{i+j=n} f_i(y) d_j(x)
\]
We take \( M = M_{1 \times 2}(\mathbb{R}) \) and \( \Gamma = \{i \in \mathbb{Z}: n \in \mathbb{Z}\} \), then \( M \) is \( \Gamma \)-ring.
We define \( D = (D_i)_{i \in \mathbb{N}} \) be a family of additive mappings of \( M \) such that \( D_n (a \ b) = (d_n(a) \ d_n(b)) \) then \( D \) is higher reverse derivation of \( M \).
Let \( F = (f_i)_{i \in \mathbb{N}} \) be a family of additive mappings of \( M \) defined by \( F_n (a \ b) = (f_n(a) \ f_n(b)) \)
Then \( F \) is a generalized higher reverse derivation of \( M \).
It is clear that every generalized higher reverse derivation of a \( \Gamma \)-ring \( M \) is Jordan generalized Higher reverse derivation of \( M \), But the converse is not true in general.

Lemma 2.3
Let \( M \) be a \( \Gamma \)-ring and let \( F = (f_i)_{i \in \mathbb{N}} \) be a Jordan generalized higher reverse derivation of \( M \) then for all \( x,y,z \in M \), \( \alpha, \beta \in \Gamma \) and \( n \in \mathbb{N} \), the following statements hold:

i) \( f_n(x \alpha y + y \alpha x) = \sum_{i+j=n} f_i(y) \alpha d_j(x) + f_i(x) \alpha d_j(y) \)
In particular if \( y \in Z(M) \)

ii) \( f_n(x \alpha y \beta x + x \beta y \alpha x) = f_n(x) \beta x \alpha y + \sum_{i+j+r=n} f_i(x) \beta d_j(y) \alpha d_r(x) + f_n(x) \alpha x \beta y + \sum_{i+j+r=n} f_i(x) \alpha d_j(y) \beta d_r(x) \)

iii) \( f_n(x \alpha y \alpha x) = f_n(x) \alpha x \alpha y + \sum_{i+j+r=n} f_i(x) \alpha d_j(y) \alpha d_r(x) \)
iv) $f_n(xayz + zayx) = f_n(z)axay + \sum_{i<n} f_i(z)ad_i(y)ad_r(x) + f_n(x)azay + \sum_{i<n} f_i(x)ad_i(y)ad_r(z)$

v) $f_n(xay\beta z) = f_n(z)bxay + \sum_{i+j+r=n} f_i(z)bd_i(y)ad_r(x)$

vi) $f_n(xay\beta z + zay\beta x) = f_n(z)bxay + \sum_{i<n} f_i(z)bd_i(y)ad_r(x) + f_n(x)bzay + \sum_{i<n} f_i(x)bd_i(y)ad_r(z)$

Proof:

i) Replace $(x + y)$ for $x$ and $y$ in definition 2.1 (i) we get:

$$f_n((x + y)\alpha(x + y)) = \sum_{i+j=n} f_i(x+y)\alpha d_i(x+y)$$

$$= \sum_{i+j=n} f_i(x)\alpha d_i(x) + f_i(y)\alpha d_i(y) + f_i(x+y)\alpha d_i(y) \quad \text{(1)}$$

On the other hand:

$$f_n((x + y)\alpha(x + y)) = f_n(x\alpha x + x\alpha y + y\alpha x + y\alpha y)$$

$$= f_n(x\alpha x + y\alpha y) + f_n(x\alpha y + y\alpha x)$$

$$= \sum_{i+j=n} f_i(x)\alpha d_i(x) + f_i(y)\alpha d_i(y) + f_i(x+y)\alpha d_i(y) \quad \text{(2)}$$

Compare (1) and (2) we get:

$$f_n(x\alpha y + y\alpha x) = \sum_{i+j=n} f_i(y)\alpha d_i(x) + f_i(x)\alpha d_i(y)$$

ii) Replacing $x\beta y + y\beta x$ for $y$ in 2.3 (i) we get:

$$f_n(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha x)$$

$$= f_n(x\alpha(x\beta y) + x\alpha(y\beta x) + (x\beta y)\alpha x + (y\beta x)\alpha x)$$

$$= f_n((x\alpha x)\beta y + (x\beta y)\alpha x + (y\beta x)\alpha x)$$

$$= \sum_{i+j=n} f_i(y)\beta d_i(x\alpha x) + f_i(x)\beta d_i(x\alpha y) + f_i(x)\alpha d_i(x\beta y) + f_i(x)\alpha d_i(y\beta x)$$

$$= \sum_{i+j+r=n} f_i(y)\beta d_i(x)\alpha d_r(x) + f_i(x)\beta d_i(y)\alpha d_r(x) + f_i(x)\alpha d_i(y)\beta d_r(x) + f_i(x)\alpha d_i(y)\beta d_r(y)$$

$$= f_n(y)bx\alpha x + \sum_{i+j+r=n} f_i(y)\beta d_i(x)\alpha d_r(x) + f_n(x)bx\alpha y + \sum_{i+j+r=n} f_i(x)\beta d_i(y)\alpha d_r(x)$$

www.iosrjournals.org 27 | Page
\[ + f_n(x) \alpha x \beta y + \sum_{i+j+r=n} f_i(x) \alpha d_j(y) \beta d_r(x) + f_n(x) \alpha \beta y x + \sum_{i+j+r=n} f_i(x) \alpha d_j(\beta) d_r(y) \ldots (1) \]

On the other hand:

\[ f_n(x \alpha y \beta y + y \beta x) = f_n(x \alpha x \beta y + x \alpha y \beta x + x \beta y \alpha x + y \beta x \alpha x) \]

\[ = f_n(y) \beta x \alpha x + \sum_{i+j+r=n} f_i(y) \beta d_j(x) \alpha d_r(\beta) + f_n(x) \alpha y \beta x + \sum_{i+j+r=n} f_i(x) \alpha d_j(\beta) d_r(y) \]

\[ + f_n(x \alpha y \beta x + x \beta y \alpha x) \ldots (2) \]

Compare (1) and (2) we get the require result.

iii) Replacing \( \alpha \) for \( \beta \) in 2.3 (ii) we have:

\[ f_n(\alpha \alpha \alpha \alpha x + \alpha x \alpha x) = 2(f_n(\alpha \alpha x)) \]

\[ = 2f_n(x) \alpha x \alpha y + \sum_{i+j+r=n} f_i(x) \alpha d_j(\alpha) d_r(\alpha) \]

Since \( M \) is 2-torsion free then we get:

\[ f_n(x \alpha y \alpha x) = f_n(x) \alpha x \alpha y + \sum_{i+j+r=n} f_i(x) \alpha d_j(\alpha) d_r(\alpha) \]

iv) Replacing \( x+z \) for \( x \) in 2.3 (iii) we have:

\[ f_n((x + y) \alpha \alpha (x + y)) = f_n(x + z) \alpha (x + z) \alpha y + \sum_{i+j+r=n} f_i(x + z) \alpha d_j(\alpha) d_r(\alpha + z) \]

\[ = f_n(x) \alpha x \alpha y + \sum_{i+j+r=n} f_i(x) \alpha d_j(\alpha) d_r(\alpha) \]

\[ + f_n(z) \alpha x \alpha y + \sum_{i+j+r=n} f_i(z) \alpha d_j(\alpha) d_r(\alpha) \]

\[ + f_n(x) \alpha z \alpha y + \sum_{i+j+r=n} f_i(x) \alpha d_j(\alpha) d_r(z) \]

\[ + f_n(z) \alpha z \alpha y + \sum_{i+j+r=n} f_i(z) \alpha d_j(\alpha) d_r(z) \ldots (1) \]

On the other hand:

\[ f_n((x + y) \alpha \alpha (x + z)) = f_n(\lambda x \lambda y + \lambda \lambda \lambda z + \lambda \lambda \lambda z) \]

\[ = f_n(x) \alpha x \lambda \alpha + \sum_{i+j+r=n} f_i(x) \alpha d_j(\lambda) d_r(\alpha) \]

\[ + f_n(z) \alpha \lambda \alpha + \sum_{i+j+r=n} f_i(z) \alpha d_j(\lambda) d_r(\lambda) \]

\[ + f_n(x) \alpha \lambda \alpha + \sum_{i+j+r=n} f_i(x) \alpha d_j(\lambda) d_r(z) \]

\[ + f_n(z) \alpha \lambda \alpha + \sum_{i+j+r=n} f_i(z) \alpha d_j(\lambda) d_r(z) + f_n(x) \alpha \lambda \alpha + \lambda \lambda \lambda \alpha \]

\[ \ldots (2) \]

Compare (1) and (2) we get the require result.
(v) Replace \((x + z)\) for \(x\) in definition 2.1(iii) we have:

\[
f_n((x + z)\alpha y)(x + z) = f_n(x + z)\beta(x + z)\alpha y + \sum_{i+j+r=n}^i i^n f_i(x + z)\beta d_i(y)\alpha d_r(x + z)
\]

\[
= f_n(x)\beta x\alpha y + \sum_{i+j+r=n}^i i^n f_i(x)\beta d_i(y)\alpha d_r(x) + f_n(z)\beta x\alpha y + \sum_{i+j+r=n}^i i^n f_i(z)\beta d_i(y)\alpha d_r(x)
\]

\[
+ f_n(z)\beta x\alpha y + \sum_{i+j+r=n}^i i^n f_i(z)\beta d_i(y)\alpha d_r(x) + f_n(z)\beta z\alpha y + \sum_{i+j+r=n}^i i^n f_i(z)\beta d_i(y)\alpha d_r(z) \ldots (1)
\]

On the other hand:

\[
f_n((x + z)\alpha y)\beta(x + z) = f_n(x\alpha y\beta x + x\alpha y\beta z + z\alpha y\beta x + z\alpha y\beta z)
\]

\[
= f_n(x\alpha y\beta x + z\alpha y\beta x + x\alpha y\beta z) + f_n(x\alpha y\beta z)
\]

\[
= f_n(x)\beta x\alpha y + \sum_{i+j+r=n}^i i^n f_i(x)\beta d_i(y)\alpha d_r(x)
\]

\[
+ f_n(z)\beta x\alpha y + \sum_{i+j+r=n}^i i^n f_i(z)\beta d_i(y)\alpha d_r(z) + f_n(z)\beta z\alpha y + \sum_{i+j+r=n}^i i^n f_i(z)\beta d_i(y)\alpha d_r(z)
\]

\[
+ f_n(x\alpha y\beta z) \ldots (2)
\]

Compare (1) and (2) we get:

\[
f_n(x\alpha y\beta z) = f_n(z)\beta x\alpha y + \sum_{i+j+r=n}^i i^n f_i(z)\beta d_i(y)\alpha d_r(x)
\]

vi) Replace \((x + z)\) for \(x\) in definition 2.1(iii) we have:

\[
f_n((x + z)\alpha y \beta(x + z)) = f_n(x + z)\beta(x + z)\alpha y + \sum_{i+j+r=n}^i i^n f_i(x + z)\beta d_i(y)\alpha d_r(x + z)
\]

\[
= (f_n(x) + f_n(z))\beta(x + z)\alpha y + \sum_{i+j+r=n}^i i^n (f_i(x) + f_i(z))\beta d_i(y)\alpha (d_r(x) + d_r(z))
\]

\[
= f_n(x)\beta x\alpha y + f_n(z)\beta x\alpha y + f_n(x)\beta z\alpha y + f_n(z)\beta z\alpha y
\]

\[
+ \sum_{i+j+r=n}^i i^n f_i(x)\beta d_i(y)\alpha d_r(x) + f_i(z)\beta d_i(y)\alpha d_r(x) + f_i(x)\beta d_i(y)\alpha d_r(z) + f_i(z)\beta d_i(y)\alpha d_r(z) \ldots .(1)
\]

On the other hand:

\[
f_n((x + z)\alpha y \beta (x + z)) = f_n(x\alpha y\beta x + x\alpha y\beta z + z\alpha y\beta x + z\alpha y\beta z)
\]

\[
= f_n(x\alpha y\beta x + z\alpha y\beta x) + f_n(x\alpha y\beta z + z\alpha y\beta z)
\]

\[
= f_n(x)\beta x\alpha y + \sum_{i+j+r=n}^i i^n f_i(x)\beta d_i(y)\alpha d_r(x)
\]

\[
+ f_n(z)\beta z\alpha y + \sum_{i+j+r=n}^i i^n f_i(z)\beta d_i(y)\alpha d_r(z) + f_n(x\alpha y\beta z + z\alpha y\beta x) \ldots .(2)
\]

Compare (1) and (2) we get the require result
Definition 2.4:
Let $F = (f_i)_{i \in \mathbb{N}}$ be a Jordan generalized higher reverse derivation of a $\Gamma$-ring $M$, then for all $x, y \in M$ and $\alpha \in \Gamma$ we define:

$$\delta_n(x, y)_\alpha = f_n(x\alpha y) - \sum_{i+j=n} f_i(y)\alpha d_j(x)$$

In the following lemma introduce some properties of $\delta_n(x, y)_\alpha$

Lemma 2.5

If $F = (f_i)_{i \in \mathbb{N}}$ is a Jordan generalized higher reverse derivation of $\Gamma$-ring $M$ then for all $x, y, z \in M$, $\alpha, \beta \in \Gamma$ and $n \in \mathbb{N}$:

i. $\delta_n(x, y)_\alpha = -\delta_n(y, x)_\alpha$

ii. $\delta_n(x + y, z)_\alpha = \delta_n(x, z)_\alpha + \delta_n(y, z)_\alpha$

iii. $\delta_n(x, y + z)_\alpha = \delta_n(x, y)_\alpha + \delta_n(x, z)_\alpha$

Proof:

i. by lemma 2.3 (i) and since $f_n$ is additive mapping of $M$ we get:

$$f_n(x\alpha y + y\alpha x) = \sum_{i+j=n} f_i(y)\alpha d_j(x) + f_i(x)\alpha d_j(y)$$

$$f_n(x\alpha y) + f_n(y\alpha x) = \sum_{i+j=n} f_i(y)\alpha d_j(x) + \sum_{i+j=n} f_i(x)\alpha d_j(y)$$

$$f_n(x\alpha y) - \sum_{i+j=n} f_i(y)\alpha d_j(x) = -f_n(y\alpha x) + \sum_{i+j=n} f_i(x)\alpha d_j(y)$$

$$f_n(x\alpha y) - \sum_{i+j=n} f_i(y)\alpha d_j(x) = -(f_n(y\alpha x) - \sum_{i+j=n} f_i(x)\alpha d_j(y))$$

$$\delta_n(x, y)_\alpha = -\delta_n(y, x)_\alpha.$$  

ii. $\delta_n(x + y, z)_\alpha = f_n((x + y)\alpha z) - \sum_{i+j=n} f_i(z)\alpha d_j(x + y)$

$$= f_n(x\alpha z + y\alpha z) - (\sum_{i+j=n} f_i(z)\alpha d_j(x) + f_i(z)\alpha d_j(y))$$

$$= f_n(x\alpha z) - \sum_{i+j=n} f_i(z)\alpha d_j(x) + f_n(y\alpha z) - \sum_{i+j=n} f_i(z)\alpha d_j(y)$$

$$= \delta_n(x, z)_\alpha + \delta_n(y, z)_\alpha.$$  

iii. $\delta_n(x, y + z)_\alpha = f_n(x\alpha (y + z)) - \sum_{i+j=n} f_i(y + z)\alpha d_j(x)$

$$= f_n(x\alpha y + x\alpha z) - \sum_{i+j=n} f_i(y)\alpha d_j(x) - f_i(z)\alpha d_j(x)$$

Since $f_n$ is additive mapping of $M$ then we have:

www.iosrjournals.org
\[\begin{align*}
&= f_n(x\alpha y) - \sum_{i+j=n} f_i(y)\alpha d_j(x) + f_n(x\alpha z) - \sum_{i+j=n} f_i(z)\alpha d_j(x) \\
&= \delta_n(x, y)_\alpha + \delta_n(x, z)_\alpha.
\end{align*}\]

iv.
\[\begin{align*}
\delta_n(x, y)_{\alpha + \beta} &= f_n(x(\alpha + \beta)y) - \sum_{i+j=n} f_i(y)(\alpha + \beta)d_j(x) \\
&= f_n(x\alpha y + x\beta y) - \sum_{i+j=n} f_i(y)\alpha d_j(x) - f_i(y)\beta d_j(x) \\
\text{Since } f_n \text{ is additive mapping} \\
&= f_n(x\alpha y) - \sum_{i+j=n} f_i(y)\alpha d_j(x) + f_n(x\beta y) - \sum_{i+j=n} f_i(y)\beta d_j(x) \\
&= \delta_n(x, y)_\alpha + \delta_n(x, y)_\beta.
\end{align*}\]

**Remark 2.6:**
Note that \( F = (f_i)_{i\in N} \) is generalized higher reverse derivation of a \( \Gamma \)-ring \( M \) if and only if \( \delta_n(x, y)_\alpha = 0 \) for all \( x, y \in M, \alpha \in \Gamma \) and \( n \in N \).

### III. The Main Results

In this section we present the main results of this paper.

**Theorem 3.1:**
Let \( F = (f_i)_{i\in N} \) be a Jordan generalized higher reverse derivation of \( M \) then \( \delta_n(x, y)_\alpha = 0 \) for all \( x, y \in M, \alpha \in \Gamma \) and \( n \in N \).

Proof:
By lemma 2.3 (i) we get:
\[f_n(x\alpha y + y\alpha x) = \sum_{i+j=n} f_i(y)\alpha d_j(x) + f_i(x)\alpha d_j(y) \ldots \ldots \ (1)\]

On the other hand:
Since \( f_n \) is additive mapping of the \( \Gamma \)-ring \( M \) we have:
\[f_n(x\alpha y + y\alpha x) = f_n(x\alpha y) + f_n(y\alpha x) = f_n(x\alpha y) + \sum_{i+j=n} f_i(x)\alpha d_j(y) \ldots \ldots \ldots \ (2)\]

Compare (1) and (2) we get:
\[f_n(x\alpha y) = \sum_{i+j=n} f_i(y)\alpha d_j(x)\]
\[f_n(x\alpha y) - \sum_{i+j=n} f_i(y)\alpha d_j(x) = 0\]

By definition 2.5 we get:
\[\delta_n(x, y)_\alpha = 0\]
Corollary 3.2:
Every Jordan generalized higher reverse derivation of \( \Gamma \)-ring \( M \) is generalized higher reverse derivation of \( M \).

Proof:

By theorem 3.1 we get \( \delta_n(x, y) = 0 \) and by Remark 2.6 we get the require result.

Proposition 3.3
Every Jordan generalized higher reverse derivation of a 2-torsion free \( \Gamma \)-ring \( M \) such that \( x\alpha yz = x\beta yz \) and \( y \in Z(M) \) is Jordan generalized triple higher reverse derivation of \( M \).

Proof:

Let \( F = (f_i)_{i \in \mathbb{N}} \) be a Jordan generalized higher reverse derivation of \( M \).

Replace \( y \) by \((x\beta y + y\beta x)\) in lemma 2.3 (i) we get

\[
f_n(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha x) = f_n((x\alpha(x\beta y) + x\alpha(y\beta x) + (x\beta y)\alpha x) + (y\beta)\alpha x)
= f_n((x\alpha x)\beta y + (x\alpha y)\beta x + (x\beta y)\alpha x + (y\beta)\alpha x)
\]

\[
= \sum_{i+j=n} f_i(y)\beta d_i(x\alpha x) + f_i(x)\beta d_j(x\alpha y) + f_i(x)\alpha d_i(y\beta x) + f_i(x)\alpha d_j(y\beta x)
\]

\[
= \sum_{i+j+n} f_i(y)\beta d_i(x)\alpha d_i(x) + f_i(x)\beta d_j(y)\alpha d_j(x) + f_i(x)\alpha d_i(y)\beta d_i(x) + f_i(x)\alpha d_j(y)\beta d_i(x)
\]

\[
= f_n(y)\beta x\alpha x + \sum_{i+j+n} f_i(y)\beta d_i(x)\alpha d_i(x) + f_n(x)\beta x\alpha y + \sum_{i+j+n} f_i(x)\beta d_i(y)\alpha d_i(x) + f_n(x)\alpha y\beta x + \sum_{i+j+n} f_i(x)\alpha d_i(y)\beta d_i(x)
\]

\[
\text{On the other hand:}
\]

\[
f_n(x\alpha(x\beta y + y\beta x) + (x\beta y + y\beta x)\alpha x) = f_n(x\alpha x\beta y + x\alpha y\beta x + x\beta y\alpha x + y\beta x\alpha x)
= f_n(x\alpha x\beta y + y\beta x\alpha x) + f_n(x\alpha y\beta x + x\beta y\alpha x)
\]

\[
= f_n(y)\beta x\alpha x + \sum_{i+j+n} f_i(y)\beta d_i(x)\alpha d_i(x)
+ f_n(x)\alpha y\beta x + \sum_{i+j+n} f_i(x)\alpha d_i(x)\beta d_i(x)
\]

\[
\text{Compare (1) and (2) and since } x\alpha yz = x\beta yz \text{ we get}
\]

\[
f_n(x\alpha y\beta x + x\alpha y\beta x) = 2(f_n(x\alpha y\beta x))
= 2(f_n(x)\beta x\alpha y + \sum_{i+j+n} f_i(x)\beta d_i(y)\alpha d_i(x))
\]

Since \( M \) is a 2-torsion free then we have:

\[
f_n(x\alpha y\beta x) = f_n(x)\beta x\alpha y + \sum_{i+j+n} f_i(x)\beta d_i(y)\alpha d_i(x)
\]
References


