Jordan (σ,τ)-Higher Homomorphisms of a Γ-Ring M into a Γ-Ring M’

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Abstract: Let M and M’ be two prime Γ-rings and σi, τi be two higher homomorphism of a Γ-ring M, for all n ∈ N in the present paper we show that under certain conditions of M, every Jordan (σ, τ)-higher homomorphism of a Γ-Ring M into a prime Γ’-Ring M’ is either (σ, τ)-higher homomorphism or (σ, τ)-anti-higher homomorphism.

Key Words: prime Γ-ring, homomorphism, Jordan homomorphism.

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I. Introduction:

Let M and Γ be two additive abelian groups, suppose that there is a mapping from M×Γ×M → M (the image of (a, a, b) being denoted by aσb, a, b ∈ M and α ∈ Γ). Satisfying for all a, b, c ∈ M and α, β ∈ Γ:

(i) \((a + b)αc = aαc + aβc\)
\(aα(b + c) = aαb + aαc\)

(ii) \((aσb)βc = aσ(bβc)\)

Then M is called a Γ-ring. This definition is due to Barnes [1], [9].

A Γ-ring M is called a prime if \(aΓMb = (0)\) implies \(a = 0\) or \(b = 0\), where \(a, b \in M\), this definition is due to [5].

A Γ-ring M is called semiprime if \(aΓMa = (0)\) implies \(a = 0\), such that \(a \in M\), this definition is due to [7].

Let M be a 2-torsion free semiprime Γ-ring and suppose that \(a, b \in M\) if \(aΓMb + bΓMa = 0\) for all \(m \in M\), then \(aΓMb = bΓMa = 0\) this definition is due to [11].

Let M be a Γ-ring then M is called 2-torsion free if \(2a = 0\) implies \(a = 0\), for every \(a \in M\), this definition is due to [6].

Let \(σ^i, τ^i\) be two higher homomorphism of a Γ-ring M then \(σ^i, τ^i\) are called commutative if \(σ^i τ^i = τ^i σ^i\), for all \(i \in \mathbb{N}\), this definition is due to Barnes [1].

Let M be a Γ-ring and \(d: M \rightarrow M\) be an additive mapping (that is \(d(a + b) = d(a) + d(b)\)) of a Γ-ring M into itself then \(d\) is called a derivation on M if:

\(d(aσb) = d(a)σb + aσd(b)\), for all \(a, b \in M\) and \(α \in Γ\).

\(d\) is called a Jordan derivation on M if \(d(aσa) = d(α)aσ + aσd(a)\), for all \(a \in M\) and \(α \in Γ\), [4], [10].

Let M be a Γ-ring and \(f: M \rightarrow M\) be an additive map (that is \(f(a + b) = f(a) + f(b)\)) Then \(f\) is called a generalized derivation if there exists a derivation \(d: M \rightarrow M\) such that \(f(aσb) = f(a)σd(b) + f(b)σd(a)\), for all \(a, b, c \in M\) and \(α \in Γ\).

And \(f\) is called a generalized Jordan derivation if there exists a Jordan derivation \(d: M \rightarrow M\) such that \(f(aσa) = f(a)σα + aσd(α)\), for all \(α \in M\) and \(α \in Γ\), [2], [3].

Let θ be an additive mapping of a Γ-ring M into a Γ-ring M’, θ is called a homomorphism if for all \(a, b \in M\) and \(α \in Γ\):

\(θ(aσb) = θ(a)σθ(b)\) [1].

And θ is called a Jordan homomorphism if for all \(a, b \in M\) and \(α \in Γ\):

\(θ(aσb + bσa) = θ(a)σθ(b) + θ(b)σθ(a)\) [8].

Let \(F\) be an additive mapping of a Γ-ring M into a Γ-ring M’, \(F\) is called a generalized homomorphism if there exists a homomorphism \(θ\) from a Γ-ring M into a Γ-ring M’, such that

\(F(aσb) = F(α)σθ(b)\), for all \(a, b \in M\) and \(α \in Γ\), where \(θ\) is called a relating homomorphism, and \(F\) is called a generalized Jordan homomorphism if there exists a Jordan homomorphism \(θ\) from a Γ-ring M into a Γ-ring M’, such that

\(F(aσb + bσa) = F(α)σθ(b) + F(b)σθ(a)\), for all \(a, b \in M\) and \(α \in Γ\), where \(θ\) is called a relating Jordan homomorphism, [8].

Let \(ζ = (ζ_i)_{i \in \mathbb{N}}\) be a family of additive mappings, \(ζ_i : M \rightarrow M’\) then \(ζ\) is said to be a higher homomorphism (resp. Jordan higher homomorphism) on a Γ-ring M into a Γ-ring M’ if \(ζ_i = I_M\) (the identity
mapping on $M'$ and $\phi_n(aab) = \sum_{i=1}^{n} \phi_i(a)\alpha\phi_i(b)$ (resp. $\phi_n(aab + b\alpha a) =$

$$\sum_{i=1}^{n} \phi_i(a)\alpha\phi_i(b) + \sum_{i=1}^{n} \phi_i(b)\alpha\phi_i(a),$$ for all $a, b \in M$ and $\alpha \in \Gamma$, [8].

Now, the main purpose of this paper is that every Jordan $(\sigma, \tau)$-higher homomorphism of a $\Gamma$-ring $M$ into a prime $\Gamma$-ring $M'$ is either $(\sigma, \tau)$-higher homomorphism or $(\sigma, \tau)$-anti-higher homomorphism and every Jordan $(\sigma, \tau)$-higher homomorphism from a $\Gamma$-ring $M$ into a 2-torsion free $\Gamma'$-ring $M'$ is a Jordan triple $(\sigma, \tau)$-higher homomorphism.

II. 2-Jordan $(\sigma, \tau)$-Higher Homomorphisms of a $\Gamma$-Rings

**Definition (2.1):**

Let $\theta = (\phi_{i,n})_{i,n}$ be a family of additive mappings of a $\Gamma$-ring $M$ into a $\Gamma$-ring $M'$ and $\sigma, \tau$ be two homomorphism of a $\Gamma$-ring $M$, $\theta$ is called a $(\sigma, \tau)$-higher homomorphism if for all $a, b \in M$, $\alpha \in \Gamma$ and $n \in N$

$$\phi_n(aab) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(b)).$$

**Definition (2.2):**

Let $\theta = (\phi_{i,n})_{i,n}$ be a family of additive mappings of a $\Gamma$-ring $M$ into a $\Gamma$-ring $M'$ and $\sigma, \tau$ be two homomorphism of a $\Gamma$-ring $M$, $\theta$ is called Jordan $(\sigma, \tau)$-higher homomorphism if for all $a, b \in M$, $\alpha \in \Gamma$ and for $n \in N$

$$\phi_n(aab + b\alpha a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(b)) + \sum_{i=1}^{n} \phi_i(\sigma^i(b))\alpha\phi_i(\tau^i(a)).$$

**Definition (2.3):**

Let $\theta = (\phi_{i,n})_{i,n}$ be a family of additive mappings of a $\Gamma$-ring $M$ into a $\Gamma$-ring $M'$ and $\sigma, \tau$ be two homomorphism of a $\Gamma$-ring $M$, $\theta$ is called Jordan triple $(\sigma, \tau)$-higher homomorphism if for all $a, b \in M$, $\alpha, \beta \in \Gamma$ and $n \in N$

$$\phi_n(aab\beta a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\sigma^{i\tau^{i-1}}(b))\beta\phi_i(\tau^i(a)).$$

**Definition (2.4):**

Let $\theta = (\phi_{i,n})_{i,n}$ be a family of additive mappings of a $\Gamma$-ring $M$ into a $\Gamma$-ring $M'$ and $\sigma, \tau$ be two homomorphism of a $\Gamma$-ring $M$, $\theta$ is called a $(\sigma, \tau)$-anti-higher homomorphism if for all $a, b \in M$, $\alpha \in \Gamma$ and $n \in N$

$$\phi_n(aab) = \sum_{i=1}^{n} \phi_i(\sigma^i(b))\alpha\phi_i(\tau^i(a)).$$

Now, we present below an example of $(\sigma, \tau)$-higher homomorphism and it is clearly is a Jordan $(\sigma, \tau)$-higher homomorphism.

**Example (2.5):**

Let $S_1, S_2$ be two rings and $\theta = (\theta_{i,n})_{i,n}$ be a $(\sigma, \tau)$-higher homomorphism of a ring $S_1$ into a ring $S_2$. let $M = \{(a,b): a, b \in S_1\}$, $M = \{(a,b): a, b \in S_2\}$ and $\Gamma = \{(n,m): n, m \in \mathbb{Z}\}$, we define $\phi = (\phi_{i,n})_{i,n}$ be a family of additive mappings from a $\Gamma$-ring $M$ into a $\Gamma$-ring $M'$, by $\phi_n((a,b)) = (\theta_{i,a}, \theta_{i,b})$ for all $a,b \in S_1$, let $\sigma_1^a, \tau_1^b$ be two homomorphisms of a $\Gamma$-ring $M$ such that

$$\sigma_1^a((a,b)) = (\sigma^a(a), \sigma^b(b)), \tau_1^a((a,b)) = (\tau^a(a), \tau^b(b))$$
then $\phi_n$ is a $(\sigma, \tau)$-higher homomorphism and Jordan $(\sigma, \tau)$-higher homomorphism of a $\Gamma$-ring $M$ into a $\Gamma$-ring $M'$.

**Lemma (2.6):**

Let $\theta = (\phi_{i,n})_{i,n}$ be a Jordan $(\sigma, \tau)$-higher homomorphism of a $\Gamma$-ring $M$ into a $\Gamma$-ring $M'$, then for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$ and $n \in N$

if $\sigma^\alpha = \sigma^i, \tau^\beta = \tau^i, \sigma^i\tau^i = \sigma^i\tau^{i-1}$ and $\sigma^i\tau^i = \tau^i\sigma^i$
\( \phi_i(a \alpha b \beta a + a \beta b \alpha a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha \phi_i(\sigma^i \tau^{n-i}(b))\beta \phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(a))\beta \phi_i(\sigma^i \tau^{n-i}(b))\alpha \phi_i(\tau^i(a)) \)

\( \phi_i(a \alpha b \beta c + c \alpha b \beta a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha \phi_i(\sigma^i \tau^{n-i}(b))\beta \phi_i(\tau^i(c)) + \sum_{i=1}^{n} \phi_i(\sigma^i(c))\alpha \phi_i(\sigma^i \tau^{n-i}(b))\beta \phi_i(\tau^i(a)) \)

In particular, if \( M' \) is a 2-torsion free commutative \( \Gamma \)-ring.

\( \phi_i(a \alpha b c) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha \phi_i(\sigma^i \tau^{n-i}(b))\beta \phi_i(\tau^i(c)) \)

\( \phi_i(a \alpha b c + c \alpha b c) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha \phi_i(\sigma^i \tau^{n-i}(b))\alpha \phi_i(\tau^i(c)) + \sum_{i=1}^{n} \phi_i(\sigma^i(c))\alpha \phi_i(\sigma^i \tau^{n-i}(b))\alpha \phi_i(\tau^i(a)) \)

**Proof:**

(i) Replace \( a\beta b + b\beta a \) for \( b \) in the definition (2.2), we get:

\( \phi_i(a \alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha \phi_i(\tau^i(a\beta b + b\beta a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(a\beta b + b\beta a))\alpha \phi_i(\tau^i(a)) \)

\[= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha \phi_i(\tau^i(a)\beta \tau^i(b) + \tau^i(b)\beta \tau^i(a)) + \]

\[= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha \phi_i(\tau^i(a)\beta \sigma^i(b) + \sigma^i(b)\beta \sigma^i(a))\alpha \phi_i(\tau^i(a)) \]

\[= \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha \left( \sum_{j=1}^{i} \phi_j(\sigma^j \tau^j(a)\beta \phi_j(\tau^j(b)) + \sum_{j=1}^{i} \phi_j(\sigma^j \tau^j(b))\beta \phi_j(\tau^j(a)) \right) \]

\[= \sum_{i=1}^{n} \left( \sum_{j=1}^{i} \phi_j(\sigma^j \tau^j(a)\beta \phi_j(\tau^j(b)) + \sum_{j=1}^{i} \phi_j(\sigma^j \tau^j(b))\beta \phi_j(\tau^j(a)) \right) \alpha \phi_i(\tau^i(a)) \]

Since \( \sigma^2 = \sigma^i, \tau^2 = \tau^i, \sigma^i \tau^i = \sigma^i \tau^{n-i} \) and \( \sigma^i \tau^i = \tau^i \sigma^i \)
\[ \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha \phi_i(\sigma^i\tau^{n-i}(a))\beta \phi_i(\tau^i(b)) + \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha \phi_i(\sigma^i\tau^{n-i}(b))\beta \phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(a))\beta \phi_i(\sigma^i\tau^{n-i}(b))\alpha \phi_i(\tau^i(a)) \]

On the other hand:
\[ \phi_n(a \alpha(a \beta b + b \beta a) + (a \beta b + b \beta a)\alpha a) = \phi_n(a \alpha a \beta b + a \alpha b \beta a + a \beta b \alpha a + b \beta a \alpha a) \]
\[ = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha \phi_i(\sigma^i\tau^{n-i}(a))\beta \phi_i(\tau^i(b)) + \sum_{i=1}^{n} \phi_i(\sigma^i(b))\beta \phi_i(\sigma^i\tau^{n-i}(a))\alpha \phi_i(\tau^i(a)) + \phi_n(a \alpha b \beta a + a \beta b \alpha a) \]

Comparing (1) and (2), we get:
\[ \phi_n(a \alpha b \beta a + a \beta b \alpha a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha \phi_i(\sigma^i\tau^{n-i}(a))\beta \phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(a))\beta \phi_i(\sigma^i\tau^{n-i}(b))\alpha \phi_i(\tau^i(a)) \]

(ii) Replace \( a + c \) for \( a \) in the definition (2.3), we get:
\[ \phi_n((a + c)\alpha b \beta (a + c)) = \sum_{i=1}^{n} \phi_i(\sigma^i(a + c))\alpha \phi_i(\sigma^i\tau^{n-i}(b))\beta \phi_i(\tau^i(a + c)) \]
\[ = \sum_{i=1}^{n} \phi_i(\sigma^i(a) + \sigma^i(c))\alpha \phi_i(\sigma^i\tau^{n-i}(b))\beta \phi_i(\tau^i(a) + \tau^i(c)) \]
\[ = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha \phi_i(\sigma^i\tau^{n-i}(b))\beta \phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha \phi_i(\sigma^i\tau^{n-i}(b))\beta \phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(c))\alpha \phi_i(\sigma^i\tau^{n-i}(b))\beta \phi_i(\tau^i(c)) \]

On the other hand:
\[ \phi_n((a + c)\alpha b \beta (a + c)) = \phi_n(a \alpha b \beta a + a \alpha b \beta c + c \alpha b \beta a + c \alpha b \beta a) \]
\[ = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha \phi_i(\sigma^i\tau^{n-i}(b))\beta \phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(c))\alpha \phi_i(\sigma^i\tau^{n-i}(b))\beta \phi_i(\tau^i(c)) + \phi_n(a \alpha b \beta c + c \alpha b \beta a) \]

Comparing (1) and (2), we get:
\[ \phi_n(a \alpha b \beta c + c \alpha b \beta a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha \phi_i(\sigma^i\tau^{n-i}(b))\beta \phi_i(\tau^i(c)) + \sum_{i=1}^{n} \phi_i(\sigma^i(c))\alpha \phi_i(\sigma^i\tau^{n-i}(b))\beta \phi_i(\tau^i(a)) \]

(iii) By (ii) since \( M' \) is a 2-torsion free commutative \( \Gamma \)-ring
\[ \phi_n(a \alpha b \beta c + c \alpha b \beta a) = 2\phi_n(a \alpha b \beta c) \]
\[ = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha \phi_i(\sigma^i\tau^{n-i}(b))\beta \phi_i(\tau^i(c)) \]

(iv) Replace \( \beta \) for \( a \) in (ii), we get:
\[ \phi_n(a \,\alpha\,b \,\alpha\,c + c\,\alpha\,b \,\alpha\,a) = \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\alpha\phi_i(\tau^i(c)) + \sum_{i=1}^{n} \phi_i(\sigma^i(c))\alpha\phi_i(\sigma^i\tau^{n-i}(b))\alpha\phi_i(\tau^i(a)) \]

**Definition (2.7):**
Let \( \theta = (\phi_n)_{n\in\mathbb{N}} \) be a Jordan \((\sigma,\tau)\)-higher homomorphism from a \( \Gamma \)-ring \( M \) into a \( \Gamma' \)-ring \( M' \); then for all \( a, b \in M, \alpha \in \Gamma \) and \( n \in \mathbb{N} \), we define \( G_n(a,b)_\alpha \) : \( M \times \Gamma \times M \longrightarrow M' \) by:

\[ G_n(a,b)_\alpha = \phi_n(a\,b\,c) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(b)) \]

Now, we present the properties of \( G_n(a,b)_\alpha \)

**Lemma (2.8):**
Let \( \theta = (\phi_n)_{n\in\mathbb{N}} \) be a Jordan \((\sigma,\tau)\)-higher homomorphism from a \( \Gamma \)-ring \( M \) into a \( \Gamma' \)-ring \( M' \), then for all \( a, b, c \in M, \alpha, \beta \in \Gamma \) and \( n \in \mathbb{N} \):

(i) \( G_n(a+b,c)_\alpha = \phi_n((a+b)\alpha\,c) - \sum_{i=1}^{n} \phi_i(\sigma^i(a+b))\alpha\phi_i(\tau^i(c)) \)

\[ = \phi_n(a\,\alpha\,c + b\,\alpha\,c) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(c)) - \sum_{i=1}^{n} \phi_i(\sigma^i(b))\alpha\phi_i(\tau^i(c)) \]

\[ = \phi_n(a\,\alpha\,c) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(c)) + \phi_n(b\,\alpha\,c) - \sum_{i=1}^{n} \phi_i(\sigma^i(b))\alpha\phi_i(\tau^i(c)) \]

\[ = G_n(a,c)_\alpha + G_n(b,c)_\alpha \]

(ii) \( G_n(a,b+c)_\alpha = \phi_n(a\,\alpha\,(b+c)) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(b+c)) \)

\[ = \phi_n(a\,\alpha\,b + a\,\alpha\,c) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(b)) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(c)) \]

\[ = \phi_n(a\,\alpha\,b) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(b)) + \phi_n(a\,\beta\,c) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\beta\phi_i(\tau^i(c)) \]

\[ = G_n(a,b)_\alpha + G_n(a,c)_\alpha \]

(iii) \( G_n(a,b)_{\alpha+\beta} = \phi_n(a\,(\alpha+\beta)b) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))(\alpha+\beta)\phi_i(\tau^i(b)) \)

\[ = \phi_n(a\,\alpha\,b) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha\phi_i(\tau^i(b)) + \phi_n(a\,\beta\,b) - \sum_{i=1}^{n} \phi_i(\sigma^i(a))\beta\phi_i(\tau^i(b)) \]

\[ = G_n(a,b)_\alpha + G_n(a,b)_\beta \]

**Remark (2.9):**
Note that \( \theta = (\phi_n)_{n\in\mathbb{N}} \) is a \((\sigma,\tau)\)-higher homomorphism from a \( \Gamma \)-ring \( M \) into a \( \Gamma' \)-ring \( M' \) if and only if \( G_n(a,b)_\alpha = 0 \) for all \( a, b \in M, \alpha \in \Gamma \) and \( n \in \mathbb{N} \).

**Lemma (2.10):**
Let $\theta = (\phi)_{cN}$ be a Jordan $(\sigma, \tau)$-higher homomorphism of a 2-torsion free $\Gamma$-ring $M$ into a $\Gamma$-ring $M'$, such that $\sigma^{n+i} = \sigma^n, \tau^n \sigma^n = \sigma^n, \sigma^i \tau^{n-i} = \tau^i \sigma^i$ and $\sigma^i \tau^i = \tau^i \sigma^i$, then for all $a, b, m \in M, \alpha, \beta \in \Gamma$ and $n \in N$

(i) $G_n(\sigma^n(a), \sigma^n(b))_{\alpha, \beta} \phi_n(\sigma^n(m)) \beta G_n(\tau^n(b), \tau^n(a))_{\alpha} +$
$G_n(\sigma^n(b), \sigma^n(a))_{\alpha} \phi_n(\sigma^n(m)) \beta G_n(\tau^n(a), \tau^n(b))_{\alpha} = 0$

(ii) $G_n(\sigma^n(a), \sigma^n(b))_{\alpha} \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(b), \tau^n(a))_{\alpha} +$
$G_n(\sigma^n(b), \sigma^n(a))_{\alpha} \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(a), \tau^n(b))_{\alpha} = 0$

(iii) $G_n(\sigma^n(a), \sigma^n(b))_{\beta} \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(b), \tau^n(a))_{\beta} +$
$G_n(\sigma^n(b), \sigma^n(a))_{\beta} \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(a), \tau^n(b))_{\beta} = 0$.

Proof:

(i) We prove by using the induction, we can assume that:
$G_s(\sigma^i(a), \sigma^i(b))_{\alpha} \beta G_s(\tau^i(b), \tau^i(a))_{\alpha} +$
$G_s(\sigma^i(b), \sigma^i(a))_{\alpha} \beta G_s(\tau^i(a), \tau^i(b))_{\alpha} = 0$ for all $a, b, m \in R$, and $s, n \in N, s < n$.

Let $w = aab \beta b \beta ha + bba \beta b \beta ab$, since $\theta$ is a Jordan $(\sigma, \tau)$-higher homomorphism $\phi_n(w) = \phi_n(aaa(b \beta b \beta b)aa + bba(\beta b \beta b)ab)$

$= \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \alpha \phi_i(\sigma^i \tau^{n-i}(b \beta b \beta b)) \alpha \phi_i(\tau^i(a)) +$
$\sum_{i=1}^{n} \phi_i(\sigma^i(b)) \alpha \phi_i(\sigma^i \tau^{n-i}(a \beta b \beta b)) \alpha \phi_i(\tau^i(b))$

$= \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \alpha \left( \sum_{j=1}^{i} \phi_i(\sigma^i \tau^{n-i}(b)) \beta \phi_i(\sigma^i \tau^{n-i}(a \beta b \beta b)) \beta \phi_i(\tau^i(\sigma^i \tau^{n-i}(b))) \right) \alpha \phi_i(\tau^i(a)) +$
$\sum_{i=1}^{n} \phi_i(\sigma^i(b)) \alpha \left( \sum_{j=1}^{i} \phi_i(\sigma^i \tau^{n-i}(a)) \beta \phi_i(\sigma^i \tau^{n-i}(b \beta b \beta b)) \beta \phi_i(\tau^i(\sigma^i \tau^{n-i}(a))) \right) \alpha \phi_i(\tau^i(b))$

$= \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \alpha \phi_i(\sigma^i \tau^{n-i}(b)) \beta \phi_i(\sigma^i \tau^{n-i}(a \beta b \beta b)) \beta \phi_i(\tau^i(\sigma^i \tau^{n-i}(b))) \alpha \phi_i(\tau^i(a)) +$
$\sum_{i=1}^{n} \phi_i(\sigma^i(b)) \alpha \phi_i(\sigma^i \tau^{n-i}(a)) \beta \phi_i(\sigma^i \tau^{n-i}(b \beta b \beta b)) \beta \phi_i(\tau^i(\sigma^i \tau^{n-i}(a))) \alpha \phi_i(\tau^i(b))$

$= \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \alpha \phi_i(\sigma^i \tau^{n-i}(b)) \beta \phi_i(\sigma^i \tau^{n-i}(a \beta b \beta b)) \beta \phi_i(\tau^i(\sigma^i \tau^{n-i}(b))) \alpha \phi_i(\tau^i(a)) +$
$\sum_{i=1}^{n} \phi_i(\sigma^i(b)) \alpha \phi_i(\sigma^i \tau^{n-i}(a)) \beta \phi_i(\sigma^i \tau^{n-i}(b \beta b \beta b)) \beta \phi_i(\tau^i(\sigma^i \tau^{n-i}(a))) \alpha \phi_i(\tau^i(b))$
Jordan \((\sigma, \tau)\)-Higher Homomorphisms of a \(\Gamma\)-Ring \(M\) into a \(\Gamma\)-Ring \(M'\)

\[
= \phi_1(\sigma^n(a))\alpha\phi_1(\sigma^n(b))\beta\phi_1(\sigma^n(m))\gamma\sum_{j=1}^{i} \phi_1(\tau^n(\sigma^n(b)))\alpha\phi(\tau^n(a)) + \\
\sum_{i=1}^{n-1} \phi_1(\sigma^n(a))\alpha\phi_1(\sigma^n(b))\beta\phi_1(\sigma^n(m))\gamma\sum_{j=1}^{i} \phi_1(\tau^n(\sigma^n(b)))\alpha\phi(\tau^n(a)) + \\
\phi_1(\sigma^n(b))\alpha\phi_1(\sigma^n(a))\beta\phi_1(\sigma^n(m))\gamma\sum_{j=1}^{i} \phi_1(\tau^n(\sigma^n(a)))\alpha\phi(\tau^n(b)) + \\
\sum_{i=1}^{n-1} \phi_1(\sigma^n(b))\alpha\phi_1(\sigma^n(a))\beta\phi_1(\sigma^n(m))\gamma\sum_{j=1}^{i} \phi_1(\tau^n(\sigma^n(a)))\alpha\phi(\tau^n(b))
\]

...(1)

On the other hand:

\[
\phi_1(w) = \phi_1((a\alpha b)\beta\alpha(\beta\alpha a) + (\beta\alpha a)\beta\alpha(\alpha\alpha b)) \\
= \sum_{i=1}^{n} \phi_1(\sigma^n(\alpha\alpha b))\beta\phi_1(\sigma^n(\beta\alpha a))\beta\phi_1(\tau^n(\beta\alpha a)) + \\
\sum_{i=1}^{n-1} \phi_1(\sigma^n(\beta\alpha a))\beta\phi_1(\tau^n(\beta\alpha a)) - \\
\phi_1(\tau^n(\beta\alpha a)) + \sum_{i=1}^{n} \phi_1(\sigma^n(\beta\alpha a))\alpha\phi_1(\tau^n(\beta\alpha a)) + \\
\sum_{i=1}^{n} \phi_1(\tau^n(\beta\alpha a))\beta\phi_1(\tau^n(\beta\alpha a))
\]

\[
= \sum_{i=1}^{n} \phi_1(\sigma^n(\alpha\alpha b))\beta\phi_1(\tau^n(\alpha\alpha b))\beta\phi_1(\tau^n(\beta\alpha a)) + \\
\sum_{i=1}^{n-1} \phi_1(\sigma^n(\alpha\alpha b))\beta\phi_1(\tau^n(\beta\alpha a)) - \\
\sum_{i=1}^{n} \phi_1(\sigma^n(\beta\alpha a))\beta\phi_1(\tau^n(\beta\alpha a)) + \\
\sum_{i=1}^{n} \phi_1(\tau^n(\beta\alpha a))\beta\phi_1(\tau^n(\beta\alpha a))
\]
\[= -\sum_{i=1}^{n} \phi_i(\sigma^i(aab))\beta \phi_i(\sigma^i \tau^{-i}(m))\beta \phi_i(\tau^i(aab)) - \sum_{j=1}^{i} \phi_j(\sigma^j \tau^{-j}(a))\alpha \phi_j(\tau^j(b)) - \sum_{i=1}^{n} \phi_i(\sigma^i(aab))\beta \phi_i(\sigma^i \tau^{-i}(m))\beta \phi_i(\tau^i(aab)) - \sum_{j=1}^{i} \phi_j(\sigma^j \tau^{-j}(a))\alpha \phi_j(\tau^j(b)) + \sum_{i=1}^{n} \phi_i(\sigma^i(a))\alpha \phi_i(\tau^i \sigma^{-i}(a))\beta \phi_i(\sigma^i \tau^{-i}(m))\beta \phi_i(\tau^i(aab)) + \sum_{i=1}^{n} \phi_i(\sigma^i(b))\alpha \phi_i(\tau^i \sigma^{-i}(a))\beta \phi_i(\tau^i(aab))
\]

\[= -\phi_n(\sigma^n(aab))\beta \phi_n((\sigma^n(m))\beta G_n(\tau^n(a), \tau^n(b))_\alpha - \sum_{i=1}^{n-1} \phi_i(\sigma^i(aab))\beta \phi_i((\sigma^i \tau^{-i}(m))\beta G_i(\tau^i(a), \tau^i(b))_\alpha - \phi_n(\sigma^n(aab))\beta \phi_n((\sigma^n(m))\beta G_n(\tau^n(b), \tau^n(a))_\alpha - \sum_{i=1}^{n-1} \phi_i(\sigma^i(aab))\beta \phi_i((\sigma^i \tau^{-i}(m))\beta G_i(\tau^i(a), \tau^i(b))_\alpha + \phi_{i+1}(\sigma^{i+1}(a))\alpha \phi_{i+1}(\tau^{i+1}(a))\beta \phi_{i+1}((\sigma^{i+1}(m))\beta \phi_{i+1}(\tau^{i+1}(aab)) + \sum_{i=1}^{n-1} \phi_i(\sigma^i(a))\alpha \phi_i(\tau^i \sigma^{-i}(a))\beta \phi_i((\sigma^i \tau^{-i}(m))\beta \phi_i(\tau^i(aab)) + \sum_{i=1}^{n-1} \phi_i(\sigma^i(b))\alpha \phi_i(\tau^i \sigma^{-i}(a))\beta \phi_i((\sigma^i \tau^{-i}(m))\beta \phi_i(\tau^i(aab))
\]

Compare (1), (2) and since \(\sigma^{n^2} = \sigma^n\), \(\tau^n \sigma^n = \sigma^n\), \(\tau^n \sigma^{-i} = \tau^i \sigma^{-i}\) and \(\sigma^i \tau^i = \tau^i \sigma^i\)

\[0 = -\phi_n(\sigma^n(aab))\beta \phi_n((\sigma^n(m))\beta G_n(\tau^n(a), \tau^n(b))_\alpha - \phi_n(\sigma^n(aab))\beta \phi_n((\sigma^n(m))\beta G_n(\tau^n(b), \tau^n(a))_\alpha - \sum_{i=1}^{n-1} \phi_i(\tau^i(\sigma^i \tau^{-i}(a)))\alpha \phi_i(\tau^i(b)) + \phi_i(b)\alpha \phi_i(\sigma^i(a))\beta \phi_i((\sigma^i(m))\beta \phi_i(\tau^i(aab)) - \sum_{i=1}^{n-1} \phi_i(\tau^i(\sigma^i \tau^{-i}(a)))\alpha \phi_i(\tau^i(b)) - \sum_{i=1}^{n-1} \phi_i(\sigma^i(aab))\beta \phi_i((\sigma^i \tau^{-i}(m))\beta G_i(\tau^i(a), \tau^i(b))_\alpha + \sum_{i=1}^{n-1} \phi_i(\sigma^i(a))\alpha \phi_i(\tau^i \sigma^{-i}(b))\beta \phi_i((\sigma^i \tau^{-i}(m))\beta \phi_i(\tau^i(aab)) - \sum_{i=1}^{n-1} \phi_i(\tau^i(\sigma^i \tau^{-i}(b)))\alpha \phi_i(\tau^i(a)) + \sum_{i=1}^{n-1} \phi_i(\tau^i(b))\alpha \phi_i(\tau^i \sigma^{-i}(a))\beta \phi_i((\sigma^i \tau^{-i}(m))\beta \phi_i(\tau^i(aab)) - \sum_{i=1}^{n-1} \phi_i(\tau^i(\sigma^i \tau^{-i}(a)))\alpha \phi_i(\tau^i(b)))
\]
\[ \begin{align*}
\phi_n(\sigma^n(ab)) & = -\phi_n(\sigma^n(ab)) \beta \phi_n((\sigma^n(m))\beta G_n(\tau^n(a), \tau^n(b)) - \\
& \quad + \phi_n(\sigma^n(a)) \alpha \phi_n(\sigma^n(b)) \beta \phi_n((\sigma^n(m))\beta G_n(\tau^n(b), \tau^n(a))
\end{align*} \]

Lemma (2.11):

Let \( \theta = (\phi_i)_{i \in \mathbb{N}} \) be a Jordan \((\sigma, \tau)\)-higher homomorphism from a \( \Gamma \)-ring \( M \) into a \( \Gamma \)-ring \( M' \), then for all \( a, b, m \in M, \alpha, \beta \in \Gamma \) and \( n \in \mathbb{N} \)

(i) \( G_n(\sigma^n(a), \sigma^n(b)) \alpha \beta \phi_n((\sigma^n(m))\beta G_n(\tau^n(b), \tau^n(a))) = G_n(\sigma^n(b), \sigma^n(a)) \alpha \beta \phi_n((\sigma^n(m))\beta G_n(\tau^n(a), \tau^n(b))) = 0 \)

(ii) \( G_n(\sigma^n(a), \sigma^n(b)) \alpha \beta \phi_n((\sigma^n(m))\alpha G_n(\tau^n(b), \tau^n(a))) = G_n(\sigma^n(b), \sigma^n(a)) \alpha \beta \phi_n((\sigma^n(m))\alpha G_n(\tau^n(a), \tau^n(b))) = 0 \)

(iii) Interchanging \( \alpha \) and \( \beta \) in (i), we get (iii).

By our hypothesis, we have:

\( G_n(\sigma^n(a), \sigma^n(b)) \alpha \beta \phi_n((\sigma^n(m))\beta G_n(\tau^n(b), \tau^n(a))) = G_n(\sigma^n(b), \sigma^n(a)) \alpha \beta \phi_n((\sigma^n(m))\beta G_n(\tau^n(a), \tau^n(b))) = 0 \)

(i) Replace \( \beta \) by \( \alpha \) in (i) proceeding in the same way as in the proof of (i) by the similar arguments, we get (ii).

(ii) Interchanging \( \alpha \) and \( \beta \) in (i), we get (iii).
(iii) \( G_n(\sigma^n(a), \sigma^n(b))_\beta \alpha \phi_n(\sigma^n(m))\alpha G_n(\tau^n(b), \tau^n(a))_\beta = 0 \)

**Proof:**

(i) By lemma (2.10) (i), we have:

\[
G_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m))\beta G_n(\tau^n(b), \tau^n(a))_\alpha + \\
G_n(\sigma^n(b), \sigma^n(a))_\alpha \beta \phi_n(\sigma^n(m))\beta G_n(\tau^n(a), \tau^n(b))_\alpha = 0
\]

And by lemma (let M be a 2-torsion free semiprime \( \Gamma \)-ring and suppose that \( a, b \in M \) if \( a\Gamma b + b\Gamma a = 0 \) for all \( m \in M \), then \( a\Gamma b = b\Gamma a = 0 \), we get:

\[
G_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m))\beta G_n(\tau^n(b), \tau^n(a))_\alpha = 0
\]

(ii) Replace \( \beta \) for \( \alpha \) in (i), we get (ii).

(iii) Interchanging \( \alpha \) and \( \beta \) in (i), we get (iii).

**Theorem (2.12):**

Let \( \theta = (\phi)_n \) be a Jordan \((\sigma, \tau)\)-higher homomorphism from a \( \Gamma \)-ring \( M \) into a prime \( \Gamma \)-ring \( M' \), then for all \( a, b, c, d, m \in M, \alpha, \beta \in \Gamma \) and \( n \in \mathbb{N} \):

(i) \( G_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m))\beta G_n(\tau^n(b), \tau^n(a) + c))_\alpha = 0 \)

(ii) \( G_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m))\beta G_n(\tau^n(d), \tau^n(c))_\alpha = 0 \)

(iii) \( G_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m))\beta G_n(\tau^n(d), \tau^n(c))_\beta = 0 \)

**Proof:**

(i) Replacing \( a + c \) for \( a \) in lemma (2.11) (i), we get:

\[
G_n(\sigma^n(a+c), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m))\beta G_n(\tau^n(b), \tau^n(a+c))_\alpha = 0
\]

\[
G_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m))\beta G_n(\tau^n(b), \tau^n(a))_\alpha + \\
G_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m))\beta G_n(\tau^n(b), \tau^n(c))_\alpha + \\
G_n(\sigma^n(c), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m))\beta G_n(\tau^n(b), \tau^n(a))_\alpha + \\
G_n(\sigma^n(c), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m))\beta G_n(\tau^n(b), \tau^n(c))_\alpha = 0
\]

By lemma (2.10)(i), we get:

\[
G_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m))\beta G_n(\tau^n(b), \tau^n(c))_\alpha + \\
G_n(\sigma^n(c), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m))\beta G_n(\tau^n(b), \tau^n(a))_\alpha = 0
\]

Therefore, we get:

\[
G_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m))\beta G_n(\tau^n(b), \tau^n(c))_\alpha \beta \phi_n(\sigma^n(m))\beta
\]

\[
= -G_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m))\beta G_n(\tau^n(b), \tau^n(c))_\alpha \beta \phi_n(\sigma^n(m))\beta
\]

\[
G_n(\sigma^n(c), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m))\beta G_n(\tau^n(b), \tau^n(a))_\alpha = 0
\]

Hence, by the primness of \( M' \):

\[
G_n(\sigma^n(a), \sigma^n(b))_\alpha \beta \phi_n(\sigma^n(m))\beta G_n(\tau^n(b), \tau^n(c))_\alpha = 0 \quad \ldots (1)
\]

Now, replacing \( b + d \) for \( b \) in lemma (2.12) (i), we get:

\[
G_n(\sigma^n(a), \sigma^n(b + d))_\alpha \beta \phi_n(\sigma^n(m))\beta G_n(\tau^n(b + d), \tau^n(a))_\alpha = 0
\]
By lemma (2.12) (i), we get:

\[ G_n(\sigma^n(a), \sigma^n(b)),_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(b), \tau^n(a))_\alpha + \]

\[ G_n(\sigma^n(a), \sigma^n(b)),_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(d), \tau^n(a))_\alpha + \]

\[ G_n(\sigma^n(a), \sigma^n(d)),_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(b), \tau^n(a))_\alpha + \]

\[ G_n(\sigma^n(a), \sigma^n(d)),_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(d), \tau^n(a))_\alpha + \]

\[ G_n(\sigma^n(a), \sigma^n(d)),_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(d), \tau^n(a))_\alpha = 0 \]

By lemma (2.12), we get:

\[ G_n(\sigma^n(a), \sigma^n(b)),_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(d), \tau^n(a))_\alpha + \]

Therefore, we get:

\[ G_n(\sigma^n(a), \sigma^n(d)),_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(b), \tau^n(a))_\alpha = 0 \]

Since \(M'\) is a prime \(\Gamma\)-ring, then:

\[ G_n(\sigma^n(a), \sigma^n(b)),_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(b + d), \tau^n(a + c))_\alpha = 0 \]

Thus, \(G_n(\sigma^n(a), \sigma^n(b)),_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(b), \tau^n(a))_\alpha + \)

\[ G_n(\sigma^n(a), \sigma^n(b)),_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(b), \tau^n(c))_\alpha + \]

\[ G_n(\sigma^n(a), \sigma^n(b)),_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(d), \tau^n(a))_\alpha + \]

\[ G_n(\sigma^n(a), \sigma^n(b)),_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(d), \tau^n(c))_\alpha = 0 \]

By (1), (2) and lemma (2.12), we get:

\[ G_n(\sigma^n(a), \sigma^n(b)),_\alpha \beta \phi_n(\sigma^n(m)) \beta G_n(\tau^n(d), \tau^n(c))_\alpha = 0 \]

By (i) and (ii), we get (ii).

(iii) Replace \(\beta\) for \(\alpha\) in (i), we get (iii).

(iii) Replacing \(\alpha + \beta\) for \(\alpha\) in (ii), we get:

\[ G_n(\sigma^n(a), \sigma^n(b)),_{\alpha + \beta} \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(d), \tau^n(c))_{\alpha + \beta} = 0 \]

\[ G_n(\sigma^n(a), \sigma^n(b)),_\alpha \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(d), \tau^n(c))_\alpha + \]

\[ G_n(\sigma^n(a), \sigma^n(b)),_\alpha \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(d), \tau^n(c))_\beta + \]

\[ G_n(\sigma^n(a), \sigma^n(b)),_\beta \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(d), \tau^n(c))_\alpha + \]

\[ G_n(\sigma^n(a), \sigma^n(b)),_\beta \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(d), \tau^n(c))_\beta = 0 \]

By (i) and (ii), we get:

\[ G_n(\sigma^n(a), \sigma^n(b)),_{\alpha + \beta} \alpha \phi_n(\sigma^n(m)) \alpha G_n(\tau^n(d), \tau^n(c))_{\alpha + \beta} = 0 \]
\[ G_n(\sigma^n(a),\sigma^n(b))_\alpha \alpha \phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\alpha + \]
\[ G_n(\sigma^n(a),\sigma^n(b))_\beta \beta \phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\beta + \]
\[ G_n(\sigma^n(a),\sigma^n(b))_\beta \beta \phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\beta + \]
\[ G_n(\sigma^n(a),\sigma^n(b))_\beta \beta \phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\beta = 0 \]

By (i) and (ii), we get:
\[ G_n(\sigma^n(a),\sigma^n(b))_\alpha \alpha \phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\alpha = 0 \]
Therefore, we have:
\[ G_n(\sigma^n(a),\sigma^n(b))_\alpha \alpha \phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\alpha = 0 \]
\[ = -G_n(\sigma^n(a),\sigma^n(b))_\alpha \alpha \phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\beta \alpha \phi_n(\sigma^n(m))\alpha \]
\[ G_n(\sigma^n(a),\sigma^n(b))_\beta \beta \phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\beta = 0 \]

Since \( M' \) is a prime \( \Gamma' \)-ring, then:
\[ G_n(\sigma^n(a),\sigma^n(b))_\alpha \alpha \phi_n(\sigma^n(m))\alpha G_n(\tau^n(d),\tau^n(c))_\alpha = 0 \].

**III. The Main Results**

**Theorem (3.1):**

Every Jordan \( (\sigma,\tau) \)-higher homomorphism from a \( \Gamma \)-ring \( M \) into a prime \( \Gamma \)-ring \( M' \) is either \( (\sigma,\tau) \)-higher homomorphism or \( (\sigma,\tau) \)-anti-higher homomorphism.

**Proof:**
Let \( \theta = (\phi_i)_{i\in\mathbb{N}} \) be a Jordan \( (\sigma,\tau) \)-higher homomorphism of a \( \Gamma \)-ring \( M \) into a prime \( \Gamma' \)-ring \( M' \).

Since \( M' \) is a prime \( \Gamma' \)-ring, we get from theorem (2.12) (i)
\[ G_n(\sigma^n(a),\sigma^n(b))_\alpha = 0 \quad \text{or} \quad G_n(\tau^n(d),\tau^n(c))_\alpha = 0, \quad \text{for all} \quad a, b, c, d \in M, \alpha, \beta \in \Gamma \quad \text{and} \quad n \in \mathbb{N}. \]

If \( G_n(\tau^n(d),\tau^n(c))_\alpha \neq 0 \) for all \( c, d \in M, \alpha \in \Gamma \) and \( n \in \mathbb{N} \), then \( G_n(\sigma^n(a),\sigma^n(b))_\alpha = 0 \) for all \( a, b \in M, \alpha \in \Gamma \) and \( n \in \mathbb{N} \), hence, we get \( \theta \) is a \( (\sigma,\tau) \)-higher homomorphism from a \( \Gamma \)-ring \( M \) into a prime \( \Gamma'-ring \) \( M' \).

**Proposition (3.2):**
Let \( \theta = (\phi_i)_{i\in\mathbb{N}} \) be a Jordan \( (\sigma,\tau) \)-higher homomorphism from a \( \Gamma \)-ring \( M \) into 2-torsion free \( \Gamma \)-ring \( M' \), such that \( ab\beta c = \beta b a \) for all \( a, b, c \in M \) and \( \alpha, \beta \in \Gamma \), \( a\alpha b\beta c = a\beta b\alpha c \), for all \( a, b, c \in M' \) and \( \alpha', \beta' \in \Gamma \), \( \sigma^2 = \sigma^{1} \), \( \tau^2 = \tau^{1} \), \( \sigma^1 \tau^1 = \sigma^1 \tau^{1} = \sigma^{1} \tau^{-1} \) and \( \sigma^1 \tau^1 = \tau^1 \sigma^1 \) then \( \theta \) is a Jordan triple \( (\sigma,\tau) \)-higher homomorphism.

**Proof:**
Replace \( b \) by \( a\beta b + b\beta a \) in the definition (2.2), we get:
\[ \phi_n(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = \]
\[ \sum_{i=1}^{n} \phi_i(\sigma^i(a)) \alpha' \phi_i(\tau^i(a\beta b + b\beta a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(a\beta b + b\beta a)\alpha \phi_i(\tau^i(a))) \]
\[ \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(b)) + \phi_i(\sigma^i(b))\phi_i(\tau^i(a)) + \phi_i(\sigma^i(a))\phi_i(\tau^i(b)) + \phi_i(\sigma^i(b))\phi_i(\tau^i(a)) + \phi_i(\sigma^i(b))\phi_i(\tau^i(a)) + \phi_i(\sigma^i(a))\phi_i(\tau^i(b)) + \phi_i(\sigma^i(b))\phi_i(\tau^i(a)) \]

Since \( aab\beta c = abbac \), for all \( a, b, c \in M' \) and \( \alpha, \beta \in \Gamma \), \( \sigma^i = \sigma^i, \tau^i = \tau^i, \sigma^i \tau^i = \sigma^i \tau^n \) and \( \sigma^i \tau^i = \tau^i \sigma^i \), we get

\[ \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\tau^i(b)) + 2 \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^n(b))\phi_i(\tau^i(a)) + \sum_{i=1}^{n} \phi_i(\sigma^i(b))\phi_i(\sigma^i\tau^n(a))\phi_i(\tau^i(a)) \]

On the other hand:
\[ \phi_n(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)a) = \phi_n(a \alpha a\beta b + a \alpha b\beta a + a\beta b a + b\beta a a a) \]

Since \( aab\beta a = abbaa \), for all \( a, b \in M \) and \( \alpha, \beta \in \Gamma \)
\[ = \phi_n(a \alpha a\beta b + b\beta a a a) + 2 \phi_n(a \alpha a\beta b) \]
\[ \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^n(a))\phi_i(\tau^i(a)) + 2 \phi_n(a \alpha a\beta b) \]

Compare (1) and (2), we get:
\[ 2 \phi_n(a \alpha a\beta b) = 2 \sum_{i=1}^{n} \phi_i(\sigma^i(a))\phi_i(\sigma^i\tau^n(b))\phi_i(\tau^i(a)) \]

Since \( M' \) is a 2-torsion free \( \Gamma \)-ring, we obtain that \( \theta \) is a Jordan triple \((\sigma, \tau)\)-higher homomorphism from a \( \Gamma \)-ring \( M \) into a \( \Gamma \)-ring \( M' \).

References:

Jordan (σ, τ)-Higher Homomorphisms of a Γ-Ring M into a Γ′-Ring M′


