Π Generalized Semi Connectedness in Intuitionistic Fuzzy Topological Spaces

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Abstract: The purpose of this paper is to introduce and study the concepts of intuitionistic fuzzy π generalized semi connectedness in intuitionistic fuzzy topological space. Some of their properties are explored.

Keywords: Intuitionistic fuzzy topology, Intuitionistic fuzzy π generalized semi closed set, Intuitionistic fuzzy π generalized semi continuous mapping, Intuitionistic fuzzy almost π generalized semi continuous mapping. Intuitionistic fuzzy π generalized semi connectedness.

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I. Introduction

The concept of fuzzy set was introduced by Zadeh in his classical paper [10] in 1965. Using the concept of fuzzy sets, Chang [2] introduced the concept of fuzzy topological space. In [1], Atanassov introduced the notion of intuitionistic fuzzy sets in 1986. Using the notion of intuitionistic fuzzy sets, Coker [3] defined the notion of intuitionistic fuzzy topological spaces in 1997. Turnali and Coke have introduced and investigated connectedness in intuitionistic fuzzy topological spaces in the year 2000. Later intuitionistic fuzzy rg-connectedness was introduced by Thakur and Rekha Chaturvedi in 2006. Recently many fuzzy topological concepts such as fuzzy connectedness have been generalized in intuitionistic fuzzy topological spaces. In this paper we have introduced intuitionistic fuzzy π generalized connectedness in fuzzy topological spaces. Also we have provided some characterizations of intuitionistic fuzzy π generalized semi connectedness.

II. Preliminaries

Definition 2.1: [1] Let X be a non empty fixed set. An intuitionistic fuzzy set (IFS in short) A in X is an object having the form $A=\{(x, \mu_A(x), \nu_A(x))/ x \in X\}$ where the functions $\mu_A(x): X \rightarrow [0, 1]$ and $\nu_A(x): X \rightarrow [0, 1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set $A$, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$. Denote by IFS(X), the set of all intuitionistic fuzzy sets in X.

Definition 2.2: [1] Let A and B be IFSs of the form $A = \{(x, \mu_A(x), \nu_A(x))/ x \in X\}$ and $B = \{(x, \mu_B(x), \nu_B(x))/ x \in X\}$. Then

(a) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$
(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$
(c) $A^c = \{(x, \mu_A(x), \nu_A(x))/ x \in X\}$
(d) $A \cap B = \{(x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x))/ x \in X\}$
(e) $A \cup B = \{(x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x))/ x \in X\}$

For the sake of simplicity, we shall use the notation $A = \langle x, \mu_A, \nu_A \rangle$ instead of $A = \{(x, \mu_A(x), \nu_A(x))/ x \in X\}$. Also for the sake of simplicity, we shall use the notation $A = \langle x, (\mu_A, \mu_B), (\nu_A, \nu_B) \rangle$ instead of $A = \langle x, (A/\mu_A, B/\mu_B) (A/\nu_A, B/\nu_B) \rangle$.

The intuitionistic fuzzy sets $0_ \ast = \{(x, 0, 1)/ x \in X\}$ and $1_ \ast = \{(x, 1, 0)/ x \in X\}$ are respectively the empty set and the whole set of X.

Definition 2.3: [3] An intuitionistic fuzzy topology (IFT in short) on a non empty X is a family $\tau$ of IFSs in X satisfying the following axioms:

(a) $0_ \ast, 1_ \ast \in \tau$
(b) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$
(c) $\cup G_i \in \tau$ for any arbitrary family $\{G_i/ i \in J\} \subseteq \tau$.

In this case the pair $(X, \tau)$ is called an intuitionistic fuzzy topological space (IFTS in short) and any IFS in $\tau$ is known as an intuitionistic fuzzy open set (IFOS in short) in X.
The complement $A^c$ of an IFOS $A$ in an IFTS $(X, \tau)$ is called an intuitionistic fuzzy closed set (IFCS in short) in $X$.

**Definition 2.4:** [3] Let $(X, \tau)$ be an IFTS and $A = \langle x, \mu_A, \nu_A \rangle$ be an IFS in $X$. Then the intuitionistic fuzzy interior and an intuitionistic fuzzy closure are defined by

$$\text{int}(A) = \bigcup \{ G / G \text{ is an IFOS in } X \text{ and } G \subseteq A \}$$

$$\text{cl}(A) = \bigcap \{ K / K \text{ is an IFCS in } X \text{ and } A \subseteq K \}.$$

Note that for any IFS $A$ in $(X, \tau)$, we have $\text{cl}(A^c) = (\text{int}(A))^c$ and $\text{int}(A^c) = (\text{cl}(A))^c$ [14].

**Definition 2.5:** An IFS $A = \langle \{ x, \mu_A(x), \nu_A(x) \} / x \in X \rangle$ in an IFTS $(X, \tau)$ is said to be an

(a) [4] intuitionistic fuzzy semi closed set (IFSCS in short) if $\text{cl}(\text{int}(A)) \subseteq A$

(b) [4] intuitionistic fuzzy $\alpha$-closed set (IF$\alpha$CS in short) if $\text{cl}(\text{int}(A)) \subseteq A$

(c) [4] intuitionistic fuzzy pre-closed set (IFPCS in short) if $\text{cl}(\text{int}(A)) \subseteq A$

(d) [4] intuitionistic fuzzy regular closed set (IFRCS in short) if $\text{cl}(\text{int}(A)) = A$

(e) [9] intuitionistic fuzzy generalized closed set (IFGCS in short) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is an IFOS

(f) [5] intuitionistic fuzzy generalized semi closed set (IF$\pi$GSCS in short) if $\text{sc}(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is an IFOS.

**Definition 2.6:** [5] An IFS $A$ is said to be an intuitionistic fuzzy $\pi$-generalized semi open set (IF$\pi$GSO in short) in $X$ if the complement $A^c$ is an IF$\pi$GSCS in $X$. The family of all IF$\pi$GSCSs of an IFTS $(X, \tau)$ is denoted by IF$\pi$GSC(X).

**Result 2.7:** [5] Every IFCS, IF$\alpha$CS, IFGCS, IFRCS, IFPCS, IF$\pi$GCS is an IF$\pi$GCS but the converses need not be true in general.

**Definition 2.8:** [6] Let $A$ be an IFS in an IFTS $(X, \tau)$. Then $\pi$-generalized Semi closure of $A$ ($\pi\text{gscl}(A)$ in short) and $\pi$-generalized Semi interior of $A$ ($\pi\text{gsint}(A)$ in short) are defined by

$$\pi\text{gscl}(A) = \bigcup \{ G / G \text{ is an IFGSOS in } X \text{ and } G \subseteq A \}$$

$$\pi\text{gsint}(A) = \bigcap \{ K / K \text{ is an IFGSCS in } X \text{ and } A \subseteq K \}.$$

**Definition 2.10:** [3] Let $f$ be a mapping from an IFS $X$ to an IFS $Y$. If $B = \langle \{ y, \mu_B(y), \nu_B(y) \} / y \in Y \rangle$ is an IFS in $Y$, then the pre-image of $B$ under $f$ denoted by $f^{-1}(B)$, is the IFS in $X$ defined by

$$f^{-1}(B) = \langle \{ x, f^{-1}(\mu_B(x)), f^{-1}(\nu_B(x)) \} / x \in X \rangle.$$

If $A = \langle \{ x, \lambda_A(x), \nu_A(x) \} / x \in X \rangle$ is an IFS in $X$, then the image of $A$ under $f$ denoted by $f(A)$ is the IFS in $Y$ defined by

$$f(A) = \langle \{ y, f(\lambda_A(x)), f(\nu_A(x)) \} / y \in Y \rangle$$

where $f(\nu_A) = 1-f(1-\nu_A)$.

**Definition 2.11:** [7] A mapping $f : (X, \tau) \to (Y, \sigma)$ is called an Intuitionistic fuzzy $\pi$-generalized semi continuous mappings, (IF$\pi$GSC continuous in short) if $f^{-1}(B)$ is an IF$\pi$GSCS in $(X, \tau)$ for every IFCS $B$ of $(Y, \sigma)$.

**Definition 2.12:** [6] A mapping $f : (X, \tau) \to (Y, \sigma)$ is called an intuitionistic fuzzy $\pi$-generalized semi irresolute (IF$\pi$GS irresolute in short) if $f^{-1}(B)$ is an IF$\pi$SGS in $(X, \tau)$ for every IF$\pi$GSCS $B$ of $(Y, \sigma)$.

**Definition 2.13:** A mapping $f : (X, \tau) \to (Y, \sigma)$ from an IFTS $(X, \tau)$ into an IFTS $(Y, \sigma)$ is said to be an

(a) [8] intuitionistic fuzzy closed mapping (IFCM for short) if $f(A)$ is an IFCS in $Y$ for every IFCS $A$ in $X$.

(b) [4] intuitionistic fuzzy semi closed mapping (IFSCM for short) if $f(A)$ is an IFCS in $Y$ for every IFCS $A$ in $X$.

(c) [4] intuitionistic fuzzy $\alpha$-closed mapping (IF$\alpha$CM for short) if $f(A)$ is an IF$\alpha$CS in $Y$ for every IFCS $A$ in $X$.

**Definition 2.14:** [5] An IFTS $(X, \tau)$ is said to be an intuitionistic fuzzy $\pi T_{1/2}$ space if every IFRWGCS in $X$ is an IFCS in $X$.

**Definition 2.15:** [5] An IFTS $(X, \tau)$ is said to be an intuitionistic fuzzy $\pi g T_{1/2}$ space if every IFRWGCS in $X$ is an IFPCS in $X$.

**Definition 2.17:** [9] An IFTS $(X, \tau)$ is said to be intuitionistic fuzzy $C_{3}$ - connected space if the only intuitionistic fuzzy sets which are both IFOS and IFCS are 0. and 1.
**III. Intuitionistic fuzzy π generalized semi connected spaces**

In this section, we have introduced intuitionistic fuzzy π generalized semi connected (IFπGS connected in short) space and studied some of its properties.

**Definition 3.1:** An IFTS $(X, \tau)$ is said to be an IFπGS connected space if the only intuitionistic fuzzy sets which are both IFGOS and IFGCS are 0- and 1-.

**Theorem 3.2:** Every IFπGS connected space is an intuitionistic fuzzy C5-connected space but not conversely.

*Proof:* Let $(X, \tau)$ be an IFπGS connected space. Suppose $(X, \tau)$ is not an intuitionistic fuzzy C5-connected space, then there exists a proper IFS $A$ which is both intuitionistic fuzzy open and intuitionistic fuzzy closed in $(X, \tau)$.

That is $A$ is both IFπGOS and IFπGCS in $(X, \tau)$. This implies that $(X, \tau)$ is not an IFπGS connected space. Therefore we get a contradiction. Hence $(X, \tau)$ must be an intuitionistic fuzzy C5-connected space.

**Example 3.3:** Let $X = \{a, b\}$ and $\tau = \{0, M, 1\}$ be an IFT on $X$, where $M = \{x, (0.4, 0.3), (0.4, 0.5)\}$. Then $(X, \tau)$ is an IFCS-connected space but not an IFπGS connected space, since the IFS $M$ in $\tau$ is both an IFπGOS and an IFπGCS in $(X, \tau)$.

**Theorem 3.4:** Every IFπGS connected space is an intuitionistic fuzzy GO-connected space but not conversely.

*Proof:* Let $(X, \tau)$ be an IFπGS connected space. Suppose $(X, \tau)$ is not an intuitionistic fuzzy GO-connected space, then there exists a proper IFS $A$ which is both IFGOS and IFGCS in $(X, \tau)$. That is $A$ is both IFπGOS and IFπGCS in $(X, \tau)$. This implies that $(X, \tau)$ is not an IFπGS connected space. That is we get a contradiction. Therefore $(X, \tau)$ must be an intuitionistic fuzzy GO-connected space.

**Example 3.5:** In Example 3.3, $(X, \sqcup)$ is an IFGO-connected space but not an IFGSP connected space.

The relation between various types of intuitionistic fuzzy connectedness is given in the following diagram.

![Diagram showing the relationship between various types of intuitionistic fuzzy connectedness](image)

The reverse implications are not true in general in the above diagram.

**Theorem 3.5:** The IFTS $(X, \tau)$ is an IFπGS connected space if and only if there exists no non-zero IFπGOS A and B in $(X, \tau)$ such that $A = B$.

*Proof: Necessity:* Let A and B be two IFπGOS in $(X, \tau)$ such that $A \neq 0$, $B \neq 0$, and $A = B$. Therefore $B'$ is an IFπGCS. Since $A \neq 0$, $B \neq 1$. This implies B is a proper IFS which is both IFπGOS and IFπGCS in $(X, \tau)$. Hence $(X, \tau)$ is not an IFπGS connected space. But it is a contradiction to our hypothesis. Thus there exists no non-zero IFπGOS A and B in $(X, \tau)$ such that $A = B$.

*Sufficiency:* Let A be both IFπGOS and IFπGCS in $(X, \tau)$ such that $0 \neq A \neq 1$. Now let $B = A^c$. Then B is an IFπGOS and $B \neq 1$. This implies $B' = A \neq 0$, which is a contradiction to our hypothesis. Therefore $(X, \tau)$ is an IFπGS connected space.

**Theorem 3.6:** Let $(X, \tau)$ be an IFπTT1/2 space, then the following are equivalent:

(i) $(X, \tau)$ is an IFπGS connected space

(ii) $(X, \tau)$ is an intuitionistic fuzzy GO-connected space

(iii) $(X, \tau)$ is an intuitionistic fuzzy C5-connected space.

*Proof: (i) $\Rightarrow$ (ii):* It is obvious from the Theorem 3.4.

(ii) $\Rightarrow$ (iii): It is obvious.

(iii) $\Rightarrow$ (i): Let $(X, \tau)$ be an intuitionistic fuzzy C5-connected space. Suppose $(X, \tau)$ is not an IFπGS connected space, then there exists a proper IFS $A$ in $(X, \tau)$ which is both IFπGOS and IFπGCS in $(X, \tau)$. But since $(X, \tau)$...
is an IFπT_{1,2} space, A is both intuitionistic fuzzy open and intuitionistic fuzzy closed in (X, τ). This implies that (X, τ) is not an intuitionistic fuzzy C3-connected, which is a contradiction to our hypothesis. Therefore (X, τ) must be an IFπGS connected space.

**Theorem 3.7:** If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is an IFπGS continuous surjection and (X, τ) is an IFπGS connected space, then (Y, σ) is an intuitionistic fuzzy C3-connected space.

**Proof:** Let (X, τ) be an IFπGS connected space. Suppose (Y, σ) is not an intuitionistic fuzzy C3-connected space, then there exists a proper IFS A which is both intuitionistic fuzzy open and intuitionistic fuzzy closed in (Y, σ). Since \( f \) is an IFπGS continuous mapping, \( f^{-1}(A) \) is both IFπGSOS and IFπGSCS in (X, τ). But it is a contradiction to our hypothesis. Hence (Y, σ) must be an intuitionistic fuzzy C3-connected space.

**Theorem 3.8:** If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is an IFπGS irresolute surjection and (X, τ) is an IFπGS connected space, then (Y, σ) is an IFπGS connected space.

**Proof:** Suppose (Y, σ) is not an IFπGS connected space, then there exists a proper IFS A such that A is both IFπGSOS and IFπGSCS in (Y, σ). Since \( f \) is an IFπGS irresolute mapping, \( f^{-1}(A) \) is both IFπGSOS and IFπGSCS in (X, τ). But this is a contradiction to our hypothesis. Hence (Y, σ) must be an IFπGS connected space.

**Definition 3.9:** An IFTS \((X, \tau)\) is an intuitionistic fuzzy C3-connected between two IFS A and B if there is no IFOS E in \((X, \tau)\) such that \(A \subseteq E\) and \(E \nsubseteq B\).

**Definition 3.10:** An IFTS \((X, \tau)\) is an IFπGS connected between two IFS A and B if there is no IFπGSOS E in \((X, \tau)\) such that \(A \subseteq E\) and \(E \nsubseteq B\).

**Example 3.11:** Let \(X = \{a, b\}\) and \(\tau = \{0., M, 1.\}\) be an IFT on X, where \(M = \{(x, 0.5, 0.3), (0.5, 0.1)\}\). Then the IFTS \((X, \tau)\) is IFπGS connected between the IFS \(A = \{(x, 0.5, 0.4), (0.5, 0.3)\}\) and \(B = \{(x, 0.5, 0.4), (0.5, 0.5)\}\).

**Theorem 3.12:** If an IFTS \((X, \tau)\) is an IFπGS connected between two IFS A and B, then it is an intuitionistic fuzzy C3-connected between A and B but the converse may not be true in general.

**Proof:** Suppose (X, τ) is not an intuitionistic fuzzy C3-connected between A and B, then there exists an intuitionistic fuzzy open set E in (X, τ) such that A ⊆ E and E ∉ B. Since every intuitionistic fuzzy open set is an IFπGSOS, there exists an IFπGSOS E in (X, τ) such that A ⊆ E and E ∉ B. This implies (X, τ) is not an IFπGS connected between A and B. That is we get a contradiction to our hypothesis. Therefore the IFTS (X, τ) must be intuitionistic fuzzy C3-connected between A and B.

**Example 3.13:** Let \(X = [a, b]\) and \(\tau = \{0., G, 1.\}\) be an IFT on X, where \(G = \{(x, 0.2, 0.2), (0.1, 0.2)\}\). Then (X, τ) is an intuitionistic fuzzy C3-connected between the IFS \(A = \{(x, 0.2, 0.3), (0.5, 0.5)\}\) and \(B = \{(x, 0.4, 0.3), (0.4, 0.5)\}\). But (X, τ) is not an IFπGS connected between A and B, since the IFS \(E = \{(x, 0.3, 0.3), (0.5, 0.4)\}\) is an IFπGSOS such that \(A \subseteq E\) and \(E \nsubseteq B\).

**Theorem 3.14:** An IFTS \((X, \tau)\) is IFπGS connected between two IFSs A and B if and only if there is no IFπGSOS and IFπGSCS E in (X, τ) such that \(A \subseteq E \subseteq B\).

**Proof** **Necessity:** Let (X, τ) be IFπGS connected between A and B. Suppose that there exists an IFπGSOS and an IFπGSCS E in (X, τ) such that \(A \subseteq E \subseteq B\), then \(E \nsubseteq B\) and \(A \subseteq E\). This implies (X, τ) is not IFπGS connected between A and B, by Definition 3.9. A contradiction to our hypothesis. Therefore there exists no IFπGSOS and an IFπGSCS E in (X, τ) such that \(A \subseteq E \subseteq B\).

**Sufficiency:** Suppose that (X, τ) is not IFπGS connected between A and B. Then there exists an IFπGSOS E in (X, τ) such that \(A \subseteq E \subseteq B\). This implies that there exists an IFπGSOS E in (X, τ) such that \(A \subseteq E \subseteq B\). But this is a contradiction to our hypothesis. Hence (X, τ) must be IFπGS connected between A and B.

**Theorem 3.15:** If an IFTS \((X, \tau)\) is IFπGS connected between A and B and \(A \subseteq A_i\), \(B \subseteq B_i\), then (X, τ) is an IFπGS connected between \(A_i\) and \(B_i\).

**Proof:** Suppose that (X, τ) is not an IFπGS connected between \(A_i\) and \(B_i\), then by Definition, there exists an IFπGSOS E in (X, τ) such that \(A_i \subseteq E \subseteq E \nsubseteq B_i\). This implies \(E \nsubseteq B_i\) and \(A_i \subseteq E\). Hence \(A \subseteq E\). Since \(E \nsubseteq B_i\), \(B_i \subseteq E\). That is \(B \subseteq B_i \subseteq E\). Hence \(E \subseteq B_i\). Therefore (X, τ) is not an IFπGS connected.
between A and B. Hence we get a contradiction to our hypothesis. Thus X must be IF\(\pi\)GS connected between \(A_1\) and \(B_1\).

**Theorem 3.16**: Let \((X, \tau)\) be an IFTS and A and B be IFS in \((X, \tau)\). If \(A \sqcap B\), then X is an IF\(\pi\)GS connected between A and B.

**Proof**: Suppose \((X, \tau)\) is not IF\(\pi\)GS connected between A and B. Then there exists an IF\(\pi\)GSOS \(E\) in \((X, \tau)\) such that \(A \subseteq E\) and \(E \subseteq B^c\). This implies that \(A \subseteq B^c\). That is \(A \sqsubseteq B\). But this is a contradiction to our hypothesis. Therefore X must be an IF\(\pi\)GS connected between A and B.

**Remark 3.17**: The converse of the above theorem may not be true in general. This can be seen from the following example.

**Example 3.18**: In Example 3.14, \((X, \tau)\) is IF\(\pi\)GS connected between the IFSs A and B but since \(\mu A(x) < \nu B(x)\), \(A \sqcap B\) is not possible.

**Theorem 3.19**: An IFTS \((X, \tau)\) is an IF\(\pi\)GS connected space if and only if there exists no non-zero IF\(\pi\)GSOS A and B in \((X, \tau)\) such that \(B = A^c\), \(B = (\text{scl}(A))^c\), \(A = (\text{scl}(B))^c\).

**Proof**: Necessity: Assume that there exists IFS A and B such that \(A \neq 0\), \(B = A^c\), \(B = (\text{scl}(A))^c\), \(A = (\text{scl}(B))^c\). Since \((\text{scl}(A))^c\) and \((\text{scl}(B))^c\) are IF\(\pi\)GSOS in \((X, \tau)\), A and B are IF\(\pi\)GSOS in \((X, \tau)\). This implies \((X, \tau)\) is not an IF\(\pi\)GS connected space, which is a contradiction. Therefore there exists no non-zero IF\(\pi\)GSOS A and B in \((X, \tau)\) such that \(B = A^c\), \(B = (\text{scl}(A))^c\), \(A = (\text{scl}(B))^c\).

Sufficiency: Let A be both IF\(\pi\)GSOS and IF\(\pi\)GSCS in \((X, \tau)\) such that \(1. \neq A \neq 0\). Now by taking \(B = A^c\), we obtain a contradiction to our hypothesis. Hence \((X, \tau)\) is an IF\(\pi\)GS connected space.

**References**