Common Fixed Point Theorem for Occasionally Converse Commuting Maps in Complex-Valued Metric Space

Rinku Sharma¹, Dinesh²
¹ Department of Mathematics, University of Delhi, Delhi 110007, India
² Department of Mathematics, D.N. College, Hisar 125001, Haryana, India

Abstract: In this paper, we proved a common fixed point theorem for occasionally converse commuting (OCC) self-maps without continuity of maps for non-complete complex-valued metric space.

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I. Introduction


A. Azam, B. Fisher and M. Khan [1] introduced the concept of complex-valued metric space and obtain a common fixed point theorem for a pair of mappings satisfying contractive type condition.

II. Definitions And Preliminaries

Let be the set of complex numbers and . Define a partial order on as follows:

\[ |z_1| < |z_2| \quad \text{if and only if} \quad \text{Re}(z_1) \leq \text{Re}(z_2), \quad \text{Im}(z_1) \leq \text{Im}(z_2). \]

Note that

\[ 0 \parallel z \Leftrightarrow |z| < 1 \]
\[ z_1 \parallel z_2, z_2 \parallel z_3 \Rightarrow z_1 \parallel z_3. \]

Definition 2.1 ([1]). Let be a nonempty set. Suppose that the mapping satisfies:

(i) \( d(x, y) = 0 \) if and only if \( x = y \);
(ii) \( d(x, y) = d(y, x) \) for all \( x, y \in X \);
(iii) \( d(x, y) \leq d(x, z) + d(z, y) \) for all \( x, y, z \in X \).

Then \( d \) is called a complex-valued metric on \( X \), and \( (X, d) \) is called a complex-valued metric space.

Example 2. Let be the set of complex numbers. Suppose that the mapping \( d : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \), defined by

\[ d(z_1, z_2) = |x_1 - x_2| + |y_1 - y_2|, \]
\[ z_1 = x_1 + iy_1, \]
\[ z_2 = x_2 + iy_2. \]

Then \( (\mathbb{C}, d) \) is a complex-valued metric space.

Definition 2.3 ([2]). A point \( x \in X \) is called a commuting point of \( f, g : X \to X \) if \( f(x) = g(x) \).

Definition 2.4 ([2]). Maps \( f, g : X \to X \) are said to be converse commuting if \( f(x) = g(x) \) implies \( f = g \).

Definition 2.5 ([6]). Two self-maps \( f, g : X \to X \) are said to be occasionally converse commuting, if for some \( x \) in \( X \) \( f(x) = g(x) \) implies \( f = g \).

Following example shows that, every conversely commuting mapping is (OCC) but the converse need not be true.
Example 2.6. Let $X = \{ z \in \mathbb{C} : \text{Im}(z) \geq 0, \text{Re}(z) = 0 \}$ and the self-mapping $f$ and $g$ are defined by:

$$f(z) = \begin{cases} 
\frac{i}{n+1}, & \text{if } z = \frac{i}{n}, n \in \mathbb{N} \\
0, & \text{otherwise} 
\end{cases}$$

and

$$g(z) = \begin{cases} 
\frac{i}{n+2}, & \text{if } z = \frac{i}{n}, n \in \mathbb{N} \\
0, & \text{otherwise} 
\end{cases}$$

Then

$$fg \left( \frac{i}{n} \right) = f \left( \frac{1}{n+2} \right) = \frac{i}{n+3}, \quad \text{and} \quad g \left( \frac{i}{n+2} \right) = g \left( \frac{i}{n+1} \right) = \frac{i}{n+3}.$$ 

But $f \left( \frac{i}{n} \right) \neq g \left( \frac{i}{n} \right)$.

III. Main Result

Theorem 3.1. Let $(X, d)$ be a complex valued-metric space and let $f, g, h$ and $k$ be four self-maps defined on $X$, such that the pairs $(f, k)$ and $(g, h)$ are occasionally converse commuting maps satisfying:

$$d(fx, gy) \leq \lambda \max \{d(kx, hy), d(fx, kx), d(gy, hy), d(gy, kx), d(fx, hy)\} \quad (3.1)$$

for all $x, y \in X$, $\lambda \in (0,1)$, then $f, g, h$ and $k$ have a unique common fixed point in $X$.

Proof. Let $OCC(f, k)$ denote the set of occasionally converse commuting points of $f$ and $k$. Since the pairs $(f, k)$ is occasionally converse commuting, by definition, there exists some $u \in OCC(f, k)$; such that $fku = kfu$ implies $fu = ku$. Hence $d(fu, ku) = 0$.

It follows that $ffu = fku$.

Similarly, the occasionally converse commuting points for the pair $(g, h)$ implies that there exists $v \in OCC(g, h)$ such that $ghv = hgv$ implies $gv = hv$. Hence $d(gv, hv) = 0$ and so $ggv = ghv$.

First, we prove that $fu = gv$. If not, then using (3.1) for $x = u, y = v$

$$d(fu, gv) \leq \lambda \max \{d(ku, hv), d(fu, ku), d(gv, hv), d(gv, ku), d(fu, hv)\}$$

which implies that $|d(fu, gv)| \leq \lambda |d(fu, gv)|$, a contradiction.

Therefore $fu = gv$.

Now, we claim that $ffu = fu$. If not, then considering (3.1) for $x = gu, y = v$, we have

$$d(ffu, gv) \leq \lambda \max \{d(kfu, hv), d(ffu, kfu), d(gv, hv), d(gv, kfu), d(ffu, hv)\}$$

which implies that $|d(ffu, fu)| \leq \lambda |d(ffu, fu)|$, a contradiction.

Therefore $ffu = fu$. Similarly $ggv = gv$. Since $fu = gv$, we have

$$fu = gv = ffu = kfu = ggv = hgv = ghv.$$ $(3.4)$

Therefore $fu = w$ (say), is a common fixed point of $f, g, h$ and $k$. For uniqueness, let $w' \neq w$ be another common fixed point of $f, g, h$ and $k$, then by (3.1), we have

$$d(fw, gfw') \leq \lambda \max \{d(kw, h'w'), d(fw, kw), d(fw', h'w'), d(gw', kw), d(fw, h'w')\}$$

which implies that $|d(w, w')| \leq \lambda |d(w, w')|$, a contradiction.

Therefore, $w = fu$ is a unique common fixed point of $f, g, h$ and $k$. 

www.iosrjournals.org 22 | Page
Corollary 3.2. Let \((X, d)\) be a complex-valued metric space and let \(f, k\) be self-maps on \(X\) such that the pair \((f, k)\) is occasionally converse commuting maps satisfying 
\[d(fx, fy) \leq \lambda \max\{d(kx, ky), d(fx, kx), d(fy, kx), d(fx, ky), d(fy, kx)\}\]
for all \(x, y \in X\), \(\lambda \in (0,1)\), then \((f, k)\) have a unique common fixed point in \(X\).

Example 3.3. Let \(X = \{z \in \mathbb{C} : 0 \leq \text{Im}(z) < 1, \text{Re}(z) = 0\}\).

Let \(d : X \times X \to \mathbb{C}\) be the metric, defined by 
\[d(z_1, z_2) = |z_1 - z_2| + |y_1 - y_2|, \text{ for all } z_1, z_2 \in \mathbb{C}\]

Define the maps \(f, g, h\) and \(k : X \to X\) as follows:
\[f(z) = \begin{cases} \frac{i}{n+4}, & \text{if } z = \frac{i}{n}, n \in \mathbb{C} \\ 0, & \text{otherwise} \end{cases}, \quad g(z) = \begin{cases} \frac{i}{n+3}, & \text{if } z = \frac{i}{n}, n \in \mathbb{C} \\ 0, & \text{otherwise} \end{cases}\]
\[h(z) = \begin{cases} \frac{i}{n+2}, & \text{if } z = \frac{i}{n}, n \in \mathbb{C} \\ 0, & \text{otherwise} \end{cases}, \quad k(z) = \begin{cases} \frac{i}{n+1}, & \text{if } z = \frac{i}{n}, n \in \mathbb{C} \\ 0, & \text{otherwise} \end{cases}\]

There exists \(u \in X - \left\{\frac{i}{n} : n \in \mathbb{C}\right\}\) such that \(fku = kfu\) implies that \(fu = ku\). Hence \((f, k)\) is (OCC). Similarly, \((g, h)\) is (OCC). The set of (OCC) of \(f\) and \(k\), and \(g\) and \(h\) are given by
\[\text{OCC}(f, k) = \text{OCC}(g, h) = u \in X - \left\{\frac{i}{n} : n \in \mathbb{C}\right\}\]

All the conditions of Theorem 3.1 are satisfied.
\[fu = ku = gv = hv = 0\] is the unique common fixed point of \(f, g, h\) and \(k\).

References