Total Dominating Sets and Total Domination Polynomials of Square of Cycles

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Abstract: Let \( G = (V, E) \) be a simple connected graph. A set \( S \subseteq V \) is a dominating set of \( G \) if every vertex is adjacent to an element of \( S \). Let \( D_i(C_n^2, i) \) be the family of all total dominating sets of the graph \( C_n^2, n \geq 6 \) with cardinality \( i \), and let \( d_i(C_n^2, i) = \left| D_i(C_n^2, i) \right| \). In this paper we construct \( d_i(C_n^2, i) \), and obtain the polynomial \( D(C_n^2, i) = \sum d_i(C_n^2, i)x^i \)
which we call total domination polynomial of \( C_n^2, n \geq 6 \) and obtain some properties of this polynomial.

Keywords: square of cycle, total domination set, total domination polynomial

I. Introduction

Let \( G = (V, E) \) be a simple connected graph. A set \( S \subseteq V \) is a dominating set of \( G \), if every vertex in \( V - S \) is adjacent to at least one vertex in \( S \). A set \( S \subseteq V \) is a total dominating set if every vertex of the graph is adjacent to an element of \( S \). The total domination number of a graph \( G \) is the minimum cardinality of a total dominating set in \( G \), and it is denoted by \( \gamma_t(G) \). Obviously \( \gamma_t(G) \leq \left| V \right| \). A cycle on \( n \) vertices is denoted by \( C_n \) which originates and concludes at the same vertex. The length of a cycle is the number of edges in the cycle. The square of a simple connected graph \( G \) is a graph with same set of vertices of \( G \) and an edge between two vertices if and only if there is a path of length at most two between them. It is denoted by \( G^2 \). We use the notation \( \lfloor x \rfloor \) for the largest integer less than or equal to \( x \) and \( \lceil x \rceil \) for the smallest integer greater than or equal to \( x \).

Also we denote the set \( \{1, 2, \ldots, n\} \) by \([n]\), throughout this paper.

Let \( C_n^2, n \geq 6 \) be the square of the cycle \( C_n, n \geq 6 \) and let \( D_i(C_n^2, i) \) be the family of total dominating sets of the graph \( C_n^2, n \geq 6 \) with cardinality \( i \) and let \( d_i(C_n^2, i) = \left| D_i(C_n^2, i) \right| \). The total domination polynomial \( D_i(C_n^2, x) \) of \( C_n^2, n \geq 6 \) is defined as \( D_i(C_n^2, x) = \sum_{i=0}^{n} d_i(C_n^2, i)x^i \), where \( \gamma_i(C_n^2) \) is the total domination number of \( C_n^2 \).

II. Total Dominating Sets Of Square Of Cycles

Let \( D_i(C_n^2, i) \) be the family of total dominating sets of \( C_n^2, n \geq 6 \) with cardinality \( i \). We will investigate total dominating sets of \( C_n^2, n \geq 6 \).

**Lemma 2.1**

\[
\gamma_i(C_n^2) = \begin{cases} \lfloor \frac{n}{5} \rfloor + 2 & \text{if } n \equiv 0 \pmod{5} \\ \lfloor \frac{n}{5} \rfloor + 1 & \text{if } n \not\equiv 0 \pmod{5} \end{cases}
\]

**Lemma 2.2**

Let \( C_n^2, n \geq 6 \) be the square of cycle \( C_n \) with \( \left| V(C_n^2) \right| = n \). Then

\( d_i(C_n^2, i) = 0 \) if \( i < \lfloor \frac{n}{5} \rfloor + 1 \) or \( i > n \) and \( d_i(C_n^2, i) > 0 \) if \( \lfloor \frac{n}{5} \rfloor + 1 \leq i \leq n \).

**Proof:**

If \( n \equiv 0 \pmod{5} \), then the total domination number of the square of cycle \( C_n^2 \) is \( \gamma_i(C_n^2) = \lfloor \frac{n}{5} \rfloor + 2 \). Therefore \( d_i(C_n^2, i) = 0 \) if \( i < \lfloor \frac{n}{5} \rfloor + 1 \) or \( i > n \).

Also \( d_i(C_n^2, i) > 0 \) if \( \lfloor \frac{n}{5} \rfloor + 1 \leq i \leq n \).

On the other hand, if \( n \not\equiv 0 \pmod{5} \), then the total domination number of \( C_n^2 \) is \( \gamma_i(C_n^2) = \lfloor \frac{n}{5} \rfloor + 1 \). Therefore \( d_i(C_n^2, i) = 0 \) if \( i < \lfloor \frac{n}{5} \rfloor + 1 \) or \( i > n \) and \( d_i(C_n^2, i) > 0 \) if \( \lfloor \frac{n}{5} \rfloor + 1 \leq i \leq n \).

Hence, in general, we have \( d_i(C_n^2, i) = 0 \) if \( i < \lfloor \frac{n}{5} \rfloor + 1 \) or \( i > n \) and \( d_i(C_n^2, i) > 0 \) if \( \lfloor \frac{n}{5} \rfloor + 1 \leq i \leq n \).
Lemma 2.3
Let \( C_n^2 \), \( n \geq 6 \) be the square of cycle with \(| V(C_n^2) | = n \). Then we have

(i) \( D_i(C_n^2, i) = \emptyset \) if \( i < \gamma_i(C_n^2) \) or \( i > n \).
(ii) \( D_i(C_n^2, x) \) has no constant term and first degree terms.
(iii) \( D_i(C_n^2, x) \) is a strictly increasing function on \([0, \infty)\).

Proof of (i)
Since \( C_n^2 \) has \( n \) vertices, there is only one way to choose all these vertices. Therefore \( d_i(C_n^2, n) = 1 \).

Out of these \( n \) vertices, every combination of \( n-1 \) vertices can dominate totally only if \( \delta(C_n^2) > 1 \).

Therefore \( D_i(C_n^2, i) = \emptyset \) if \( i < \gamma_i(C_n^2) \) and \( D_i(C_n^2, n+k) = \emptyset \), \( k = 1, 2, 3, \ldots \).

Thus we have \( d_i(C_n^2, i) = 0 \) for \( i < \gamma_i(C_n^2) \) and \( d_i(C_n^2, n+i) = 0 \), for \( i = 1, 2, 3, \ldots \).

Proof of (ii)
A single vertex of \( C_n^2 \) cannot totally dominate all the vertices of \( C_n^2 \) \( n \geq 6 \). So the set of all vertices of \( C_n^2 \) is totally dominated at least two of the vertices of \( C_n^2 \). Hence the total domination polynomial has no constant term as well as first degree term.

Proof of (iii)
By the definition of total domination, every vertex of \( C_n^2 \) is adjacent to an element of total dominating set. That is \( D_i(C_n^2, x) = \sum_{i=0}^{n} D_i(C_n^2, n) x^i \).
Therefore \( D_i(C_n^2, x) \) is a strictly increasing function on \([0, \infty)\).

Lemma 2.4
Let \( C_n^2 \), \( n \geq 6 \) be the square of cycle with \(| V(C_n^2) | = n \). Then we have

(i) If \( D_i(C_{n-2}^2, i-1) = D_i(C_{n+3}^2, i-1) = \emptyset \), then \( D_i(C_n^2, i-1) = \emptyset \).
(ii) If \( D_i(C_{n-2}^2, i-1) \neq \emptyset \) and \( D_i(C_{n+3}^2, i-1) \neq \emptyset \), then \( D_i(C_n^2, i-1) \neq \emptyset \).
(iii) If \( D_i(C_{n+1}^2, i-1) = D_i(C_{n-1}^2, i-1) = \emptyset \), \( D_i(C_{n+3}^2, i-1) = \emptyset \), \( D_i(C_{n-3}^2, i-1) = \emptyset \), \( D_i(C_{n+2}^2, i-1) = \emptyset \) and \( D_i(C_{n-2}^2, i-1) = \emptyset \), then \( D_i(C_n^2, i) = \emptyset \).

Proof of (i)
Since \( D_i(C_{n+1}^2, i-1) = \emptyset \) and \( D_i(C_{n-1}^2, i-1) = \emptyset \),
\[ d_i(C_{n+1}^2, i-1) = 0 \text{ and } d_i(C_{n-1}^2, i-1) = 0. \]

Then \( i-1 < \left\lfloor \frac{n-1}{5} \right\rfloor + 1 \) or \( i-1 > n-1 \) and \( i-1 < \left\lfloor \frac{n-3}{5} \right\rfloor + 1 \) or \( i > n-3 \).

If \( i < \frac{n-3}{5}+1 \), then \( i-1 < \frac{n-2}{5} \).

Therefore \( D_i(C_{n+2}^2, i-1) = \emptyset \).
If \( i > n-1 \), then \( i-1 > n-2 \).
Therefore \( D_i(C_{n-2}^2, i-1) = \emptyset \).
Hence in all the cases \( D_i(C_{n-2}^2, i-1) = \emptyset \).

Proof of (ii)
Since \( D_i(C_{n+1}^2, i-1) \neq \emptyset \) and \( D_i(C_{n-1}^2, i-1) \neq \emptyset \),
\[ d_i(C_{n+1}^2, i-1) \neq 0 \text{ and } d_i(C_{n-1}^2, i-1) \neq 0. \]

Then \( \left\lfloor \frac{n-3}{5} \right\rfloor + 1 \leq i-1 \leq n-1 \) and \( \left\lfloor \frac{n-3}{5} \right\rfloor + 1 \leq i-1 \leq n-3 \).

\[ \Rightarrow \left\lfloor \frac{n-3}{5} \right\rfloor + 1 \leq i-1 \leq n-3, \text{ since } \left\lfloor \frac{n-1}{5} \right\rfloor + 1 \leq i-1. \]

\[ \Rightarrow \left\lfloor \frac{n-2}{5} \right\rfloor + 1 \leq i-1 \leq n-2. \]

\[ \Rightarrow d_i(C_{n-2}^2, i-1) \neq 0. \]
\[ \Rightarrow D_i(C_{n-2}^2, i-1) \neq \emptyset. \]

Proof of (iii)
Since \( D_i(C_{n+1}^2, i-1) = \emptyset \), \( D_i(C_{n-1}^2, i-1) = \emptyset \), \( D_i(C_{n+3}^2, i-1) = \emptyset \), \( D_i(C_{n+3}^2, i-1) = \emptyset \) and \( D_i(C_{n+3}^2, i-1) = \emptyset \).
\[ d_i(C_{n-3}^2, i-1) = 0, d_i(C_{n+2}^2, i-1) = 0, d_i(C_{n-2}^2, i-1) = 0 \text{ and } d_i(C_{n+3}^2, i-1) = 0. \]

\[ \Rightarrow i-1 < \frac{n-3}{5} + 1 \text{ or } i > n-1; \]
\[ i-1 < \frac{n-2}{5} + 1 \text{ or } i > n-2; \]
If \( i \) \( < \frac{n-3}{5} + 1 \), or \( i > n - 3 \), or \( i > n - n \), then we have
\[
D_i(C_n, i) = \varnothing.
\]

Therefore, \( D_i(C_n, i) = \varnothing \).

If \( i > n - 1 \), then \( i > n - 2 \), then \( i > n - 3 \), then \( i > n - 4 \) and \( i > n - 5 \).

\[
D_i(C_n, i) = \varnothing.
\]

Lemma 2.5

Let \( C_n \), \( n \geq 6 \) be the square of cycle with \( |V(C_n^2)| = n \). Suppose that \( D_i(C_n, i) \neq \varnothing \), then we have
\[
\begin{align*}
(i) & \quad D_i(C_n, i) = \varnothing, \quad D_i(C_n, i) = \varnothing, \quad D_i(C_n, i) = \varnothing, \quad D_i(C_n, i) = \varnothing, \quad \text{if and only if} \quad n = i. \\
(ii) & \quad D_i(C_n, i) = \varnothing, \quad D_i(C_n, i) = \varnothing, \quad D_i(C_n, i) = \varnothing, \quad D_i(C_n, i) = \varnothing, \quad \text{if and only if} \quad n = i - 2. \\
(iii) & \quad D_i(C_n, i) = \varnothing, \quad D_i(C_n, i) = \varnothing, \quad D_i(C_n, i) = \varnothing, \quad D_i(C_n, i) = \varnothing, \quad \text{if and only if} \quad n = i - 3.
\end{align*}
\]

Proof of (i)

Suppose
\[
D_i(C_n, i) = \varnothing, \quad D_i(C_n, i) = \varnothing, \quad D_i(C_n, i) = \varnothing, \quad D_i(C_n, i) = \varnothing, \quad \text{if and only if} \quad n = i.
\]

If \( i > n - 1 \), then \( i > n - 2 \), then \( i > n - 3 \), then \( i > n - 4 \), then \( i > n - 5 \).

\[
D_i(C_n, i) = \varnothing.
\]

Proof of (ii)

Suppose
\[
D_i(C_n, i) = \varnothing, \quad D_i(C_n, i) = \varnothing, \quad D_i(C_n, i) = \varnothing, \quad D_i(C_n, i) = \varnothing, \quad \text{if and only if} \quad n = i.
\]
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D_4(C_{n+2},i-1) \neq \varnothing \text{ and } D_4(C_{n+2},i-1) = \varnothing \\
\implies d_t(C_{n+2},i-1) \neq 0, d_t(C_{n+2},i-1) \neq 0, d_t(C_{n+2},i-1) \neq 0, \\
\text{and } d_t(C_{n+2},i-1) = 0 \\
\implies [\frac{n-1}{5}] + 1 \leq i-1 \leq n-1 ; [\frac{n-2}{5}] + 1 \leq i-1 \leq n-2 ; \\
[\frac{n-3}{5}] + 1 \leq i-1 \leq n-3 ; [\frac{n-4}{5}] + 1 \leq i-1 \leq n-4 \text{ and } \\
i-1 < [\frac{n-5}{5}] + 1 \text{ or } i-1 > n-5 \\
\implies [\frac{n-5}{5}] + 1 \leq [\frac{n-4}{5}] + 1 \leq i-1 \\
\implies d_t(C_{n+2},i-1) \neq 0 \text{ which is a contradiction} \because \text{ since } d_t(C_{n+2},i-1) = 0. \\
\text{Therefore } i-1 < [\frac{n-5}{5}] + 1 \text{ is not possible, so } i-1 > n-5 \\
\implies i > n-4 \\
\implies i \geq n-3 \\
\text{since } d_t(C_{n+2},i-1) \neq 0 \\
\implies [\frac{n-3}{5}] + 1 \leq i-1 \leq n-4 \\
\implies i-1 \leq n-4 \\
\implies i \leq n-3 \\
\text{Hence } i = n-3 \\
\text{Conversely, if } i = n-3, \text{ then } \\
D_4(C_{n-1},i-1) = D_t(C_{n-1},n-4) = \varnothing \\
D_t(C_{n+2},i-1) = D_t(C_{n+2},n-4) = \varnothing \\
D_t(C_{n+3},i-1) = D_t(C_{n+3},n-4) = \varnothing \\
D_t(C_{n+4},i-1) = D_t(C_{n+4},n-4) = \varnothing \\
\text{But } D_t(C_{n+5},i-1) = D_t(C_{n+5},n-4) = \varnothing \\
\textbf{Proof of (iii)} \\
\text{Suppose,} \\
D_t(C_{n+2},i-1) \neq \varnothing, D_t(C_{n+2},i-1) \neq \varnothing, D_t(C_{n+3},i-1) \neq \varnothing, D_t(C_{n+4},i-1) \neq \varnothing \text{ and } \\
D_t(C_{n+5},i-1) \neq \varnothing \\
\implies d_t(C_{n+2},i-1) \neq 0, d_t(C_{n+2},i-1) \neq 0, d_t(C_{n+3},i-1) \neq 0, \\
d_t(C_{n+4},i-1) \neq 0 \text{ and } d_t(C_{n+5},i-1) \neq 0 \\
\implies [\frac{n-1}{5}] + 1 \leq i-1 \leq n-1 ; [\frac{n-2}{5}] + 1 \leq i-1 \leq n-2 ; \\
[\frac{n-3}{5}] + 1 \leq i-1 \leq n-3 ; \\
[\frac{n-4}{5}] + 1 \leq i-1 \leq n-4 \text{ and } [\frac{n-5}{5}] + 1 \leq i-1 \leq n-5 \\
\implies [\frac{n-1}{5}] + 1 \leq i-1 \leq n-5 \\
\implies [\frac{n-5}{5}] + 2 \leq i \leq n-4. \\
\text{Conversely, if } [\frac{n-1}{5}] + 2 \leq i \leq n-4. \\
\implies [\frac{n-1}{5}] + 1 \leq i-1 \leq n-5 \\
\implies [\frac{n-5}{5}] + 1 \leq [\frac{n-2}{5}] + 1 \leq [\frac{n-3}{5}] + 1 \leq [\frac{n-4}{5}] + 1 \leq [\frac{n-5}{5}] \leq i-1 \leq n-4 \text{ if } 3 < n < 5 \leq n-3 < 5 < n-1 \\
\implies [\frac{n-3}{5}] + 1 \leq i-1 \leq n-5 ; [\frac{n-5}{5}] + 1 \leq i-1 \leq n-4 ; [\frac{n-4}{5}] + 1 \leq i-1 \leq n-3 ; \\
[\frac{n-2}{5}] + 1 \leq i-1 \leq n-2 \text{ and } [\frac{n-1}{5}] + 1 \leq i-1 \leq n-1 \\
\implies D_t(C_{n-1},i-1) \neq \varnothing, D_t(C_{n-2},i-1) \neq \varnothing, D_t(C_{n-3},i-1) \neq \varnothing, D_t(C_{n-4},i-1) \neq \varnothing \text{ and } \\
D_t(C_{n-5},i-1) \neq \varnothing. \\
\textbf{Theorem 2.6} \\
\text{For every } n \geq 8 \text{ and } i > [\frac{n}{5}] + 1, \text{ then we have} \\
(i) \quad D_t(C_{k \cdot 2^k} \cdot 2k) = \{1,3,\ldots,7k-6,7k-4\}, \{1,6,\ldots,7k-6,7k-1\}, \{2,4,\ldots,7k-5,7k-3\}, \\
\{2,7,\ldots,7k-5,7k\}, \{3,5,\ldots,7k-4,7k-2\}, \{4,6,\ldots,7k-3,7k-1\}, \\
\{5,7,\ldots,7k-2,7k\}\} \text{ for } k \geq 1. \\
(ii) \quad \text{If } D_t(C_{n+2},i-1) = D_t(C_{n+3},i-1) = D_t(C_{n+4},i-1) = D_t(C_{n+5},i-1) = \varnothing \text{ and}
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\[ D_n(C_{n+1}^2, i-1) \neq \emptyset \text{ then } D_n(C_{n+2}^2, i) = \{n\} \]

(iii) If \( D_n(C_{n+1}^2, i-1) \neq \emptyset, D_n(C_{n+2}^2, i-1) \neq \emptyset \) and \( D_n(C_{n+3}^2, i-1) \neq \emptyset \), then \( D_n(C_{n+2}^2, i) = \{n\} \cdot \{x \in \mathbb{E} \} = \{n\} \)

Proof of (i):

For any \( k \geq 1 \), split the vertices of \( C_{n+2}^2 \) into \( k \) number of sets of the form \( \{1,2,3,4,5,6,7\}, \{8,9,10,11,12,13,14\}, \ldots, \{7k-6,7k-5,7k-4,7k-3,7k-2,7k-1,7k\} \). The seven total dominating sets of cardinality \( 2k \) are constructed by choosing first or third or first or second and fourth or second and seventh or third or fifth or fourth and sixth or fifth and seventh from each set. Hence \( D_n(C_{n+2}^2, x) \) has the only seven total dominating sets such as \( \{1,3, \ldots, 7k-6, 7k-4\} \).

Proof of (ii):

Since \( D_n(C_{n+1}^2, i-1) = D_n(C_{n+2}^2, i-1) = D_n(C_{n+3}^2, i-1) = \emptyset \) and \( D_n(C_{n+2}^2, i) = \emptyset \), then by Lemma 2.5(i), we have, \( \{n\} \cdot \{x \in \mathbb{E} \} = \{n\} \).

Proof of (iii):

Since \( D_n(C_{n+1}^2, i-1) \neq \emptyset, D_n(C_{n+2}^2, i-1) \neq \emptyset \) and \( D_n(C_{n+3}^2, i-1) \neq \emptyset \), then \( D_n(C_{n+2}^2, i) = \emptyset \).

By Lemma 2.5(ii), we have, \( \{n\} \cdot \{x \in \mathbb{E} \} = \{n\} \).

Theorem 2.7

For every \( n \geq 6 \) and \( i \geq \lceil \frac{n}{5} \rceil + 1 \) if \( D_n(C_{n+1}^2, i-1) \neq \emptyset, D_n(C_{n+2}^2, i-1) \neq \emptyset \), then \( D_n(C_{n+2}^2, i) = \{X \cup \{n\} / X \in D_n(C_{n+1}^2, i) \cup \{X \cup \{n-3\} / X \in D_n(C_{n+3}^2, i) \cup \{X \cup \{n-4\} / X \in D_n(C_{n+2}^2, i) \} \}

Proof of (2.8):

If \( D_n(C_{n+2}^2, i) \) is the family of the total dominating sets of \( C_n^2 \) with cardinality \( i \), where \( i \geq \lceil \frac{n}{5} \rceil + 1 \), then \( d_n(C_{n+2}^2, i) = d_n(C_{n+1}^2, i-1) + d_n(C_{n+2}^2, i-1) + d_n(C_{n+3}^2, i-1) + d_n(C_{n+2}^2, i-1) \)

Proof:

By theorem (2.6) and (2.7).

(i) If \( D_n(C_{n+1}^2, i-1) = D_n(C_{n+2}^2, i-1) = D_n(C_{n+3}^2, i-1) = \emptyset \) and \( D_n(C_{n+2}^2, i) = \{n\} \), then \( D_n(C_{n+2}^2, i-1) = \{n\} \).

(ii) If \( D_n(C_{n+1}^2, i-1) \neq \emptyset, D_n(C_{n+2}^2, i-1) \neq \emptyset \) and \( D_n(C_{n+3}^2, i-1) \neq \emptyset \), then \( D_n(C_{n+2}^2, i) = \{n\} \cdot \{x \in \mathbb{E} \} = \{n\} \).

(iii) \( D_n(C_{n+2}^2, i) = \{X \cup \{n\} / X \in D_n(C_{n+1}^2, i) \cup \{X \cup \{n-1\} / X \in D_n(C_{n+2}^2, i) \cup \{X \cup \{n-3\} / X \in D_n(C_{n+3}^2, i) \} \}

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By this construction in each case, we obtain that
\[ |D_t(C_n^2,i)| = |D_t(C_{n-2}^2,i-1)| + |D_t(C_{n-3}^2,i-1)| + |D_t(C_{n-4}^2,i-1)| \]
Therefore \( d_t(C_n^2,i) = d_t(C_{n-2}^2,i-1) + d_t(C_{n-3}^2,i-1) + d_t(C_{n-4}^2,i-1) \)

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III. Total Domination Polynomial Of Square Of Cycles

Let \( D_t(C_n^2,x) = \sum_{i=\gamma(C_n^2)}^n d_t(C_n^2,x)x^i \), be the total domination polynomial of the \( C_n^2 \), \( n \geq 6 \).

In this section we study this polynomial.

**Theorem 3.1**

For every \( n \geq 6 \) and \( i > \left\lceil \frac{n}{2} \right\rceil \), we have
\[
D_t(C_n^2,x) = xD_t(C_{n-2}^2,x) + D_t(C_{n-3}^2,x) + D_t(C_{n-4}^2,x) \]
with the initial values
\[
D_t(C_n^2,x) = 12x^2 + 20x^3 + 15x^4 + 6x^5 + x^6,
\]
\[
D_t(C_n^2,x) = 7x^7 + 28x^8 + 35x^9 + 21x^{10} + 7x^{11} + x^{12},
\]
\[
D_t(C_n^2,x) = 24x^3 + 62x^4 + 56x^5 + 28x^6 + 8x^7 + x^8,
\]
\[
D_t(C_n^2,x) = 9x^9 + 81x^{10} + 117x^{11} + 84x^{12} + 36x^9 + x^9.
\]

**Proof**

If \( i > \left\lceil \frac{n}{2} \right\rceil \) and by theorem 2.5 and 2.6, we have
\[
D_t(C_n^2,i) = 12x^2 + 20x^3 + 15x^4 + 6x^5 + x^6,
\]
\[
D_t(C_n^2,i) = 7x^7 + 28x^8 + 35x^9 + 21x^{10} + 7x^{11} + x^{12},
\]
\[
D_t(C_n^2,i) = 24x^3 + 62x^4 + 56x^5 + 28x^6 + 8x^7 + x^8,
\]
\[
D_t(C_n^2,i) = 9x^9 + 81x^{10} + 117x^{11} + 84x^{12} + 36x^9 + x^9.
\]

By theorem 3.1, we obtain \( d_t(C_n^2,i) \), for \( 6 \leq n \leq 14 \) as shown in Table 2.1. There are interesting relationship between numbers in this Table. In the following theorem we obtain some properties of \( d_t(C_n^2,i) \).

**Theorem 3.2**

The following properties hold for the coefficients of \( D_t(C_n^2,x) \):

(i) \( d_t(C_n^2,n) = 1 \), for every \( n \geq 6 \)

(ii) \( d_t(C_n^2,n-1) = n \), for every \( n \geq 6 \)

(iii) \( d_t(C_n^2,n-2) = \frac{1}{2}(n(n-1)) \), for every \( n \geq 6 \)

(iv) \( d_t(C_n^2,n-3) = \frac{1}{6}(n(n-1)(n-2)) \), for every \( n \geq 6 \)

(v) \( d_t(C_n^2,n-4) = \frac{1}{24}(n(n-1)(n-2)(n-3)) \), for every \( n \geq 7 \)

(vi) \( d_t(C_n^2,2k) = 7 \), for every \( k \geq 1 \)

**Proof**

(i) Since for any graph \( G \) with \( n \) vertices, \( d_t(G,n) = 1 \), then \( d_t(C_n^2,n) = 1 \).

(ii) Since \( D_t(C_n^2,n-1) = |\{x\} / x \in [n]| \)
Total Dominating Sets and Total Domination Polynomials of Square of Cycles

\[ D_t(C_2(n-1)) = nC_1 + d_t(C_2(n-1)) = n \]

(iii) To prove \( d_t(C_3(n-2)) = \frac{1}{24} n(n-1) \)

We apply induction on \( n \).

When \( n = 6 \)

L.H.S = \( d_t(C_3, 4) = 15 \) (from table)

R.H.S = \( \frac{1}{24} [6(6-1)] = 15 \)

Therefore the result is true for \( n = 6 \).

Suppose that the result is true for all natural numbers less than \( n \), and we prove it for \( n \).

We have \( d_t(C_3(n-2)) = d_t(C_3(n-1), 3) + d_t(C_3(n-1), 4) + d_t(C_3(n-3), 3) + d_t(C_3(n-3), 4) + d_t(C_3(n-4), 4) \)

\[ = \frac{1}{6} [n(n-1)(n-2)] + \frac{1}{2} [(n-1)(n-2)(n-3)] + n - 2 + 1 + 0 \]

\[ = \frac{1}{2} [n^2 - 3n + 2 + 2n - 2] \]

\[ = \frac{1}{2} (n^2 - n) \]

\[ = \frac{1}{2} n(n-1) \), for every \( n \geq 6 \)

Hence the result is true for all \( n \).

Hence by induction hypothesis, we have

\[ d_t(C_3(n-2)) = \frac{1}{2} n(n-1) \], for every \( n \geq 6 \)

(iv) To prove \( d_t(C_3(n-3)) = \frac{1}{6} n(n-1)(n-2) \), for every \( n \geq 6 \)

We apply induction on \( n \).

When \( n = 6 \)

L.H.S = \( d_t(C_3, 3) = 20 \) (from table)

R.H.S = \( \frac{1}{6} [6(6-1)(6-2)] = 20 \)

Therefore the result is true for \( n = 6 \).

Suppose that the result is true for all natural numbers less than \( n \), and we prove it for \( n \).

We have \( d_t(C_3(n-3)) = d_t(C_3(n-1), 4) + d_t(C_3(n-1), 4) + d_t(C_3(n-3), 4) + d_t(C_3(n-3), 4) + d_t(C_3(n-4), 4) \)

\[ = \frac{1}{6} [n(n-1)(n-2)(n-3)] + \frac{1}{2} [(n-1)(n-2)(n-3)] + n - 2 \]

\[ = \frac{1}{6} [n^2 - 3n + 6n - 6 + 2n - 2] \]

\[ = \frac{1}{6} [n^2 - 2n + 2n] \]

\[ = \frac{1}{6} n(n-1)(n-2) \) for \( n \geq 6 \)

Hence the result is true for all \( n \).

Hence by induction hypothesis, we have

\[ d_t(C_3(n-3)) = \frac{1}{6} n(n-1)(n-2) \], for every \( n \geq 6 \)

(v) To prove \( d_t(C_3(n-4)) = \frac{1}{24} n(n-1)(n-2)(n-3) - n \), for every \( n \geq 7 \)

We apply induction on \( n \).

When \( n = 7 \)

L.H.S = \( d_t(C_3, 3) = 28 \) (from table)

R.H.S = \( \frac{1}{24} [7(7-1)(7-2)(7-3)] - 28 \)

Therefore the result is true for \( n = 6 \).

Suppose that the result is true for all natural numbers less than \( n \), and we prove it for \( n \).

We have \( d_t(C_3(n-4)) = d_t(C_3(n-1), 5) + d_t(C_3(n-1), 5) + d_t(C_3(n-3), 5) + d_t(C_3(n-3), 5) + d_t(C_3(n-4), 5) \)

\[ = \frac{1}{24} [n(n-1)(n-2)(n-3)(n-4)] - (n-1) + \frac{1}{6} [(n-2)(n-3)(n-4)] + \frac{1}{2} [(n-3)(n-4)] + (n-4) \]

\[ = \frac{1}{24} [n^2 - 3n + 12n - 12] + \frac{1}{6} [n^2 - 7n + 12] + \frac{1}{2} (n^2 - 7n + 12) + (n-4) \]

\[ = \frac{1}{24} [n^2 - 7n + 12n^2 - 3n^2 + 21n^2 - 36n + 2n^2 - 14n + 24 - 24n + 24 + 4n^2 - 28n^2 + 48n - 8n^2 + 56n - 96 + 128n - 144 + 24n - 96] \]

\[ = \frac{1}{24} [n^2 - 6n^2 + 11n^2 - 30n] \]

\[ = \frac{1}{24} [n^2 - 6n^2 + 11n^2 - 6n - 24n] \]

\[ = \frac{1}{24} [n^2 - 6n^2 + 11n^2 - 6n] \]

\[ = \frac{1}{24} [n(n-1)(n-2)(n-3)] - n \]

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Hence the result is true for all \( n' \).

Hence by induction hypothesis, we have

\[
\Delta_t(C_n^2, n-4) = \frac{1}{24} \{n(n-1)(n-2)(n-3)\} - n, \text{ for every } n \geq 7
\]

(vi) To prove \( \Delta_t(C_7^2, 2k) = 7 \), for every \( k \geq 1 \)

By theorem (2.6)(i)

\[
\Delta_t(C_7^2, 2k) = \{1, 3, \ldots, 7k-6, 7k-4\} \cup \{1, 6, \ldots, 7k-5, 7k-3\} \\
\{2, 7, \ldots, 7k-5, 7k\} \cup \{3, 5, \ldots, 7k-4, 7k-2\} \cup \{4, 6, \ldots, 7k-3, 7k-1\} \\
\{5, 7, \ldots, 7k-2, 7k\} \text{ for } k \geq 1.
\]

Therefore, \( |\Delta_t(C_7^2, 2k)| = 7 \)

Hence \( \Delta_t(C_7^2, 2k) = 7 \), for every \( k \geq 1 \)

Theorem 3.3

(i) For every \( n \geq 10 \) and \( i > \left\lceil \frac{n}{5} \right\rceil + 1 \), we have

\[
\Delta_t(C_n^2, j+1) = \Delta_t(C_n^2, j) + \Delta_t(C_n^2 - 1, j) + \Delta_t(C_n^2 - 2, j) + \Delta_t(C_n^2 - 3, j)
\]

Proof:

By theorem (2.8)

We have

\[
\Delta_t(C_n^2, j+1) = \Delta_t(C_n^2, j) + \Delta_t(C_n^2 - 1, j) + \Delta_t(C_n^2 - 2, j) + \Delta_t(C_n^2 - 3, j)
\]

or

\[
\Delta_t(C_n^2, j+1) = \Delta_t(C_n^2, j) + \Delta_t(C_n^2 - 1, j) + \Delta_t(C_n^2 - 2, j) + \Delta_t(C_n^2 - 3, j)
\]

IV. Conclusion

We obtain total domination sets and total domination polynomial square of cycles. Similarly we can find total domination sets and total domination polynomial of specified graphs.

References


