On Pairwise Completely Regular Ordered Spaces

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Abstract: In this paper, we introduce the concept of pairwise 0-completely regular filters on a pairwise completely regular ordered bitopological space; we define the category of bitopological ordered K-spaces, which is isomorphic to that found among both bitopological and ordered spaces.

Key words & Phrases: A bitopological ordered space; a bitopological partially ordered space; a pairwise completely regular ordered space; a pairwise 0-completely regular filter; pairwise continuous isotone; a pairwise compact ordered space; pairwise Gc-set; pairwise k-compact; pairwise k-Lindelöf; bitopological P-spaces

I. Introduction

In (1963) Kelly, J. C. [2] initiated the study of bitopological spaces. A set equipped with two topologies is called a bitopological space. Since several others authors continued investigating such spaces; among them recently [4]. In (1965) Nachbin, L. [6] initiated the study of topological ordered spaces. A topological ordered space is a triple (X, τ, ≤), where τ is a topology on X and ≤ is a partial order on X. In (1971) Singal, M. K. and Singal, A. R. [9] introduced the concept of a bitopological ordered space, and they studied some separation axioms for such spaces. Raghavan, T. G. [7, 8] and various other authors have contributed to development and construction some properties of such spaces. In (1976) Choe, T. H. and Hong, Y. H. [1] introduced the concept of 0-completely regular filters on a completely regular ordered space and gave some results with this concept. Kopperman, R. and Lawson, J. D. [5] defined bitopological and topological ordered K-spaces in order to handle the requirements of domain theory in theoretical computer science. The aim of this paper is to introduce the concept of pairwise 0-completely regular filters on a pairwise completely regular ordered bitopological space; we define the category of bitopological ordered K-spaces, which is isomorphic to that found among both bitopological and ordered spaces.

II. Preliminaries and notations

Let (X, ≤) be a partially ordered set (i.e. a set X together with a reflexive, antisymmetric and transitive relation ≤). For a subset A ⊆ X, we write:

L(A) = {y ∈ X: y ≤ x for some x ∈ A}, and
M(A) = {y ∈ X: x ≤ y for some x ∈ A}.

In particular, if A is a singleton set, say {x}, then we write L(x) and M(x) respectively. A subset A of X is said to be decreasing (resp. increasing) if A = L(A) (resp. A = M(A)). The complement of a decreasing (resp. an increasing) set is an increasing (resp. a decreasing) set. A mapping f: (X, ≤) → (X', ≤') from a partially ordered set (X, ≤) to a partially ordered set (X', ≤') is increasing (resp. a decreasing) if x ≤ y in X implies f(x) ≤'f(y) (resp. f(y) ≤'f(x)). f is called isotone if its monotone increasing and therefore order-preserving. f is called an order isomorphism if it is an increasing bijection such that f⁻¹ is also increasing.

A bitopological ordered space [9] is a quadruple consisting of a bitopological space (X, τ₁, τ₂), and a partial order ≤ on X; it is denoted as (X, τ₁, τ₂, ≤). The partial order ≤ is said to be continuous (resp. weakly continuous) [7] if its graph G(≤) = {(x, y): x ≤ y} is closed in the product topology τ₁ × τ₂ (resp. τ₁ × τ₂) where i, j = 1, 2; i ≠ j, or equivalently, if L(x) and M(x) are τᵢ-closed, where i, j = 1, 2 (resp. L(x) is τᵢ-closed and M(x) is τᵢ-closed), for each x ∈ X. If the bitopological space equipped with (weakly) continuous partial order, then the space is (weakly) pairwise Hausdorff. If a bitopological space (X, τ₁, τ₂) is equipped with a continuous partial order ≤, then (X, τ₁, τ₂, ≤) will be called a bitopological partially ordered space.

For a subset A of a bitopological ordered space (X, τ₁, τ₂, ≤),

\[ H_{\tau_1}^i(A) = \bigcap \{ F \mid F \text{ is } \tau_1 \text{-decreasing closed subset of } X \text{ containing } A \}, \]

\[ H_{\tau_1}^m(A) = \bigcap \{ F \mid F \text{ is } \tau_1 \text{-increasing closed subset of } X \text{ containing } A \}, \]

Clearly, \( H_{\tau_1}^m(A) \) (resp. \( H_{\tau_1}^i(A) \)) is the smallest \( \tau_1 \)-increasing closed set containing A. Further A is \( \tau_1 \)-increasing (resp. \( \tau_1 \)-decreasing) closed if and only if \( A = H_{\tau_1}^m(A) \) (resp. \( H_{\tau_1}^i(A) \)).

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A map $f: (X, \tau_1, \tau_2, \leq) \rightarrow (X', \tau_{*1}, \tau_{*2}, \leq')$ is pairwise continuous isotone if $f: (X, \tau_1, \tau_2) \rightarrow (X', \tau_{*1}, \tau_{*2})$ is pairwise continuous and $f: (X, \leq) \rightarrow (X', \leq')$ is isotone.

The category of bitopological ordered spaces and pairwise continuous isotone functions will be denoted by BTOS. Let $(X, \tau_1, \tau_2, \leq)$ be a bitopological ordered space. Let $\mathfrak{F}$ (resp. $\mathfrak{R}$) be a filter in $X$ consisting of $\tau_i$-decreasing (resp. $\tau_i$-increasing) closed subsets of $X$ where $i, j = 1, 2; i \neq j$. A pair $(\mathfrak{F}, \mathfrak{R})$ of $\mathfrak{F}$, $\mathfrak{R}$ is called a pairwise filter on $X$ if $\forall F \subseteq \mathfrak{F}$ and $G \subseteq \mathfrak{R}$.

For given two $(\mathfrak{F}, \mathfrak{R}) \subseteq (\mathfrak{F}_1, \mathfrak{R}_1)$, $(\mathfrak{F}, \mathfrak{R}_2)$ and $(\mathfrak{F}_2, \mathfrak{R}_2)$ we defined a relation $(\mathfrak{F}, \mathfrak{R}_1) \subseteq (\mathfrak{F}_2, \mathfrak{R}_2)$ if and only if $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$ and $\mathfrak{R}_1 \subseteq \mathfrak{R}_2$, we can easily remark by Zorn’s Lemma that every pairwise filter is contained in a maximal pairwise filter.

III. Pairwise completely regular ordered spaces

Let $I$ denote $[0, 1]$, considered as a set $([0, 1], \sigma, \omega)$; as a bitopological space, where $\sigma = \{ (a, 1) \mid a \in [0, 1] \} \cup \{ [0, 1]\}$, and $\omega = \{ [0, a] \mid a \in [0, 1]\} \cup \{ [0, 1]\}$.

Definition 3.1: A bitopological partially ordered space $(X, \tau_1, \tau_2, \leq)$ will be called pairwise completely separated provided that whenever $x \leq y$ in $X$, there exists a pairwise continuous isotone $f: X \rightarrow I$ such that $f(x) > f(y)$ where $I$ is the unit interval with the usual order and bitopology.

The category of bitopological partially ordered spaces and pairwise continuous isotone functions will be denoted by BTPOS.

Definition 3.2: A pairwise completely separated bitopological partially ordered space $(X, \tau_1, \tau_2, \leq)$ is said to be a pairwise completely regular ordered space if for any point $x \in X$ and for any $\tau_i$-open neighborhood $V$ of $x$ there exists continuous isotone $f: X \rightarrow I$ and a pairwise continuous ant-isotone $g: X \rightarrow I$ such that $f(x)=1$ and $g(x)=0$ for each $r=1, 2, \ldots$, and $u^f_i$ (resp. $u^g_i$) is contained in $\cup f^r_i((-1, 1))$, either $i, j = 1, 2$; $i \neq j$.

By a maximal pairwise 0-completely regular filter on $X$ is meant a pairwise 0-completely regular filter not contained in any other pairwise 0-completely regular filter.

Remark 3.4: For every pairwise 0-completely regular filter, there exists by Zorn’s Lemma, a maximal pairwise 0-completely regular filter containing it. In particular, a pairwise 0-completely regular filter $(\mathfrak{F}, \mathfrak{R})$ on a pairwise completely regular ordered space $X$ is a maximal pairwise 0-completely regular filter iff for any pair of $\tau_i$-(resp. $\tau_j$-) open sets $U$ and $V$ with $V \subseteq U$ and finitely many pairwise continuous isotones $f_1, f_2, \ldots, f_n: X \rightarrow [-1, 1]$ such that $f_i(V) = 0$ for each $i$.

Theorem 3.5: A pairwise filter $(\mathfrak{F}, \mathfrak{R})$ on a pairwise completely regular ordered space $(X, \tau_1, \tau_2, \leq)$ is said to be 0-completely regular ordered space $X$ such that for every point $x \in X$ and for any $\tau_i$-open neighborhood $V$ of $x$ there exists a pairwise continuous isotone $f: X \rightarrow I$ and a pairwise continuous ant-isotone $g: X \rightarrow I$ such that $f(x)=1$ and $g(x)=0$ for each $r=1, 2, \ldots$, and $u^f_i$ (resp. $u^g_i$) is contained in $\cup f^r_i((-1, 1))$, either $i, j = 1, 2$; $i \neq j$.

Proof. Since pairwise filter containing a convergent pairwise filter is a gain convergent, its enough to show that every maximal pairwise 0-completely regular filter satisfies the necessary conditions. Let $(\mathfrak{F}, \mathfrak{R})$ be a maximal pairwise 0-completely regular filter on $X$ and $f$ a member of pairwise hom $(X, [-1, 1])$.

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The following definition of pairwise compactness is due to Kim [3].

Definition 4.1: Let \( (X, \tau_1, \tau_2, \leq) \) be a bitopological ordered space. Let \( \tau(i, V) = \{ \emptyset, X, \{ U \cup V \mid U \in \tau_i \} \} \) where \( V \in \tau_i, i = 1, 2; i \neq j \). If \( \tau(i, V) \) is compact for every \( V \in \tau_i \) then the space is called pairwise compact.

Definition 4.2: A pairwise compact bitopological space \( (X, \tau_1, \tau_2) \) equipped with a continuous partial order is called a pairwise compact ordered space.

Note that, if a continuous partial order is replaced by a weakly continuous partial order in the above definition, then we obtain on the definition of a pairwise G- compact space due to Raghavan, T. G. [8].

The proofs of the two following lemmas are similar to which in [7]

Lemma 4.3: If \( (X, \tau_1, \tau_2) \) is bitopological space equipped with a continuous partial order, then if K is \( \tau_1 \)-closed (resp. \( \tau_2 \)-closed) then \( L(K) \) (resp. \( M(K) \) ) is \( \tau_1 \)-closed (resp. \( \tau_2 \)-closed) closed, i, j = 1, 2; i \neq j.

Lemma 4.4: Let \( (X, \tau_1, \tau_2, \leq) \) be a pairwise compact ordered space. If \( P \) is an increasing \( \tau_1 \)-closed (resp. a decreasing \( \tau_2 \)-closed) neighborhood of \( P \), then there exists an increasing \( \tau_1 \)-closed (resp. a decreasing \( \tau_2 \)-closed) neighborhood of \( P \).

Definition 4.5: Let \( k \) be an infinite cardinal. A pairwise completely regular ordered space \( (X, \tau_1, \tau_2, \leq) \) is called pairwise \( k \)-compact if every maximal pairwise \( 0 \)-regular filter on \( X \) with the \( k \)-intersection property is convergent.

Definition 4.6 [1]: Let \( k \) be an infinite cardinal, and let \( (X, \tau) \) be a topological space. A subset of \( X \) is called a \( G_k \)-set if it is an intersection of fewer than \( k \)-open subsets of \( X \). A subset of \( X \) is called \( k \)-closed if it is closed with respect to the topology by all \( G_k \)-sets of \( X \).

Let \( (X, \tau_1, \tau_2) \) be a bitopological space, \( \mathbb{A} \subseteq X \), we say that \( A \) is a pairwise \( G_k \)-set if \( A \) is \( G_k \)-set with respect to both \( \tau_1 \) and \( \tau_2 \). A subset of \( X \) is called pairwise closed if it is closed with respect to both \( \tau_1 \) and \( \tau_2 \).

For a pairwise completely regular ordered space \( (X, \tau_1, \tau_2, \leq) \). Let \( \beta X \) be the set of all maximal pairwise \( 0 \)-regular filters on \( X \), endowed with the topologies \( \tau(i, V) \) (resp. \( \tau_1 \)-open) generated by \( \{ X \} \cup \{ U \mid U \in \beta X \} \) where \( U \in \beta X \) (resp. \( U \in \tau_1 \)). U is an increasing \( \tau_1 \)-closed (resp. a decreasing \( \tau_2 \)-closed) open set, i, j = 1, 2; i \neq j. And a relation \( \leq^* \) defined as follows: (\( \mu_1, U_1 \)) \( \leq^* \) (\( \mu_2, U_2 \)) in \( \beta X \) if \( f(\mu_1) \leq f(\mu_2) \) and \( f(U_1) \leq f(U_2) \) for all \( f \in \text{hom}(X, [-1, 1]) \) at 0.

Lemma 4.7: The space \( (\beta X, \leq, \tau^*, \tau^*_2) \) is a bitopological partially ordered space.

Proof. We wish to prove that the partial order \( \leq \) is continuous in the product \( \tau_1 \times \tau_2 \), i, j = 1, 2; i \neq j. Let \( (\mu_1, U_1) \) and \( (\mu_2, U_2) \) are two elements of \( \beta X \) with \( (\mu_1, U_1) \leq (\mu_2, U_2) \) in \( \beta X \), and we wish to find \( \tau_1 \)-increasing (resp. \( \tau_2 \)-decreasing) neighborhood of \( (\mu_1, U_1) \), (resp. \( (\mu_2, U_2) \)). Since \( (\mu_1, U_1) \leq (\mu_2, U_2) \) implies to \( \mu_1 \leq \mu_2 \) and \( U_1 \leq U_2 \), there are \( f \in \text{hom}(X, [-1, 1]) \) with \( f(\mu_1) \leq f(\mu_2) \) and \( f(U_1) \leq f(U_2) \).
Let $r_1$ and $r_2$ be elements of $[-1, 1]$ with $\lim \left( \mu_2, U_2 \right) < r_1 < r_2 < \lim \left( \mu_1, U_1 \right)$ and let $U = f^{-1}([-1, r_1[ \text{ and } V = f^{-1}([r_2, 1])$. Then it is obvious that $U'$ (resp. $V'$) is a $\tau_j^-$ (resp. $\tau_j^+$) neighborhood of $\left( \mu_1, U_1 \right)$ (resp. $\left( \mu_2, U_2 \right)$) and that for any $\left( I_1, R_1 \right) \in V'$ and any $\left( I_2, R_2 \right) \in U'$, $\left( I_1, R_1 \right) \leq \left( I_2, R_2 \right)$; i.e. $U'$ (resp. $V'$) is an increasing (resp. decreasing) as desired.

Lemma 4.8: A pairwise completely regular ordered space $(X, \tau_1, \tau_2, \leq)$ is called a pairwise k-compact if and only if it is pairwise k-closed in $\beta X$.

Proof. Noting that $\beta X$ is the strict extension of $X$ with all pairwise 0-completely regular filters as the pairwise filter trace, the proof is immediate from Lemma 4.3.

For an infinite cardinal $k$, the category of pairwise k-compact ordered spaces and pairwise continuous isotones will be denoted by PKCOS.

Definition 4.9 [1]: Let $k$ be an infinite cardinal. A Hausdorff space is said to be $k$-Lindelöf if every filter with the k-intersection property has a cluster point.

Definition 4.10: Let $(X, \tau_1, \tau_2)$ be a bitopological space. Let $\tau(i, V) = \{ \sigma, X, \{ U \cup V \mid U \in \tau_i \} \}$ where $V \in \tau_j$, $i, j = 1, 2; i \neq j$. If $\tau(i, V)$ is $k$-Lindelöf for every $V \in \tau_j$, then the space is called pairwise k-Lindelöf.

The notion of P-space is well known. A topological space is called a P-space if and only if every $G_\delta$-set is open. A space $(X, \tau_1, \tau_2)$ is called a bitopological P-space [8] if and only if $(X, \tau_1)$ and $(X, \tau_2)$ are both P-spaces.

Definition 4.11: A bitopological P-orderd space $(X, \tau_1, \tau_2, \leq)$ is called pairwise k-ordered Lindelöf if and only if the $(X, \tau_1, \tau_2)$ is pairwise k-Lindelöf and the partial order $\leq$ is continuous.

Proposition 4.12: A pairwise k-Lindelöf completely regular ordered space $(X, \tau_1, \tau_2)$ is pairwise k-compact ordered space.

Proof. For any maximal pairwise 0-completely regular filter $(\mathcal{I}, \mathcal{R})$ on $X$ with the k-intersection property, let $x$ (resp.y) be a cluster point of $\mathcal{I}$ (resp. $\mathcal{R}$). Then $N(x) \cap I \cap N(y) \cap R$ exists; $N(x) = \mathcal{I}$ (resp. $N(y) = \mathcal{R}$). Hence $(\mathcal{I}, \mathcal{R})$ is convergent as desired.

Proposition 4.13: If a pairwise filter $(\mathcal{I}, \mathcal{R})$ on a pairwise completely regular ordered space $(X, \tau_1, \tau_2, \leq)$ contains a maximal pairwise 0-completely regular filter with the countable intersection property then $f(\mathcal{I})$ (resp. $f(\mathcal{R})$) is convergent for any pairwise continuous isotope $f: X \rightarrow R$.

Proof. It is enough to show that for any maximal pairwise 0-completely regular filter $(\mathcal{I}, \mathcal{R})$ with the countable intersection property and a pairwise continuous isotope $f: X \rightarrow R$, $f(\mathcal{I})$ (resp. $f(\mathcal{R})$) is convergent. Since $f(\mathcal{I})$ (resp. $f(\mathcal{R})$) is a filter $\tau_i$- (resp. $\tau_j$-) base with the countable intersection property and $R$ is pairwise Lindelöf, $f(\mathcal{I})$ (resp. $f(\mathcal{R})$) has a cluster point. Moreover, by the same argument as that in the proof of theorem 3.4, one can easily show that $f(\mathcal{I})$ (resp. $f(\mathcal{R})$) has only one cluster point and that $f(\mathcal{I})$ (resp. $f(\mathcal{R})$) converges to the point.

References