A New Bound for the Gamma Function
In the Direction of W.K.Hayman

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Abstract: In this paper I have extended a result of W.K.Hayman to Euler’s gamma function which is known to be a logarithmically Convex function.

Key Words: Euler’s gamma function, convex function.

I. Introduction And Main Results

By Bohr – Mollerup – Artin theorem, we can define the Euler gamma function as follows.

**Theorem A:** Let \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) satisfy each of the following properties.

i) \( \log f(x) \) is a convex function

ii) \( f(x+1) = x f(x) \quad \forall \ x \in \mathbb{R}^+ \)

iii) \( f(1) = 1 \)

Then, \( f(x) = \Gamma(x), \quad \forall \ x \in \mathbb{R}^+ \)

But, Usually the Euler Gamma function is introduced as a function of a real variable and is defined via an integral

\[
\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \ x>0
\]

Here, we can observe that \( \Gamma(x+1) = x! \)

One can easily verify the following properties.

**Property 1 :**

\( \Gamma(x) \) is a logarithmically convex function

(or)

\( f(x) = \log \Gamma(x) \) is a convex function.

The proof of this property follows directly from the following definition.

**Definition :**

Let \( \Phi(x) \) be a real valued function on \([a, b]\) and let

\( \Phi''(x) \geq 0 \) for all \( x \in [a, b]. \) Or, Equivalently, let \( \Phi' \) be non decreasing on \([a, b]. \) Then, \( \Phi \) is a convex function on \([a, b]. \)

Or \( \Phi \) is said to be convex on \([a, b] \) iff

\[
|\Phi(x_1) - \Phi(x_2)| x_1 - x_2 \leq 0
\]

i.e. \( \Phi(x_1) (x_3 - x_2) + \Phi(x_2) (x_2 - x_1) \geq \Phi(x_3) (x_3 - x_1) \quad (1) \)

Clearly, one can observe that \( \Phi(x) = \log \Gamma(x) \) satisfies (1) and hence \( \log \Gamma(x) \) is a convex function on \([a, b]. \)

We wish to establish the following theorem.

**Theorem :**

Suppose that \( g(x) = \log \Gamma(x) \) is a strictly increasing and convex function of \( x \) for \( x > x_0. \) Then given \( k > 1 \) there exists a sequence \( x_n \rightarrow \infty \) such that if \( f(x) \) is any other positive increasing and convex function of \( x \) such that \( f(x) < g(x) \) for \( x \geq x_0, \) then we have,

\[
\Gamma(x_n)f^n(x_n) < e^k \Gamma(x_n) \quad (n=1,2, \ldots)
\]
Here, \( f'(x) \) denotes the right derivative of \( f(x) \) and \( \frac{\Gamma'(x)}{\Gamma(x_n)} \) is the left derivative of \( g(x) \).

To prove the above result, we require the following lemma [Hayman].

**Lemma** [1]: Suppose that \( \Phi(x) \) is positive for \( x > x_0 \) and bounded in every interval \([x_0, x_1]\) when \( x_0 < x_1 < \infty \). Then given \( k > 1 \) there exists a sequence \( x_n \to \infty \) such that

\[
\Phi(x) < k \frac{\Phi(x_n)}{x_n} \text{ for } x_n < x < x_n + \frac{1}{\log^k(\Phi(x_n))} + \frac{1}{\Phi(x_n)}
\]

**Proof of the theorem:**

Since \( g(x) = \log \Gamma(x) \) is convex, \( g'(x) \) is non-decreasing. Since \( g(x) \) is strictly increasing, \( g'(x) > 0 \) for \( x > x_0 \). Also \( g'(x) \) is bounded above in any finite interval \((x_0, x_1)\) for \( x_1 > x_0 \). Thus, we may apply the above lemma to the function \( \Phi(x) = \frac{g'(x)}{g(x)} \) and hence we can find a sequence \( x_n \to \infty \) such that.

\[
\Phi(x) < k \frac{\Phi(x_n)}{x_n} \text{ for } x_n < x < x_n + \frac{1}{\Phi(x_n)}
\]

Also, if \( x_n^1 = x_n + \frac{1}{\Phi(x_n)} \), we have,

\[
\log g(x_n^1) - \log g(x_n) = \int_{x_n}^{x_n^1} \frac{g'(x)}{g(x)} \, dx < (x_n^1 - x_n) k \Phi(x_n) = k.
\]

Hence, \( g(x_n^1) < e^k g(x_n) \)

Then, since \( f(x) \) is increasing, we have

\[
f'(x_0) \leq \frac{1}{x_n^1 - x_n} \int_{x_n}^{x_n^1} f'(x) \, dx = \frac{f(x_n^1) - f(x_n)}{x_n^1 - x_n} \leq \Phi(x_n) \frac{f(x_n^1)}{x_n^1 - x_n}
\]

\[
\leq \Phi(x_n) g(x_n^1)
\]

\[
= e^k \Phi(x_n) g(x_n)
\]

Thus, \( f'(x_0) \leq e^k g(x_n) \)

Or \( f'(x_n) \leq e^k \frac{\Gamma'(x_n)}{\Gamma(x_n)} \)

Or \( \Gamma(x_n) f'(x_n) < e^k \Gamma'(x_n) \) \( (n = 1, 2, 3 \ldots) \)

**References**
