Models of Finsler Spaces With Given Geodesics

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Abstract: In the present paper, we introduce the theory of four dimensional Finsler space and define geodesic equation with the basis of Finsler space. We also try to define geodesic equation to useful significance.

I. Introduction

Finsler geometry is a kind of differential geometry, which was originated by P. Finsler in 1918. It is usually considered as a generalization of Riemannian geometry. The definition of finsler space-

1.1 Finsler Space:

Suppose that we are given a function L(x^i, y^j) of the line element (x^i, y^j) of a curve defined in R. We shall assume L as a function of at least C^1 in all its 2n-arguments. If we define the infinitesimal distance ds between two points P(x^i) and Q(x^i + dx^i) of R by the relation

\[ ds = L(x^i, dx^i) \]

(1.1.1)

then the manifold M^n equipped with the fundamental function L defining the metrix (1.1.1) is called a Finsler space. If L(x^i, dx^i) satisfies the following conditions.

Condition A-

The function L(x^i, y^j) is positively homogeneous of degree one in y^i i.e.

\[ L(x^i, ky^j) = k L(x^i, y^j), k > 0 \]

(1.1.2)

Condition B-

The function L(x^i, y^j) is positively if not all y^i vanish simultaneously i.e.

\[ L(x^i, y^j) > 0 \] with \[ \sum_i(y^i)^2 \neq 0 \]

(1.1.3)

Condition C-

The quadratic form

\[ \partial_i L^2(x, y) \epsilon^i \epsilon^j = \frac{\partial^2 L^2(x, y)}{\partial y^i \partial y^j} \epsilon^i \epsilon^j \]

(1.1.4)

is assumed to be positive definite for any variable \( \epsilon^i \).

Form Euler’s theorem on homogenous functions, we have

\[ \partial_i \partial_j L^2(x, y) y^i y^j = L(x, y) \]

(1.1.5)

\[ \partial_i \partial_j L^2(x, y) y^i = 0 \]

(1.1.6)

We put

\[ g_{ij}(x, y) = \frac{1}{2} \partial_i \partial_j L^2(x, y) \]

(1.1.7)

Using the theory of quadratic form and the condition C, we deduce form (1.1.4) that-

\[ g(x, y) = g_{ij}(x, y) > 0 \]

(1.1.8)

for all line elements (x^i, y^j). If the function L is of particular form

\[ L(x^i, dx^i) = [g_{ij}(x^i) dx^i dx^j]^{1/2} \]

(1.1.9)

where the coefficients g_{ij}(x^i) are independent of dx^i, the metric defined by this function is called Riemannian metric and manifold M^n is called a Riemannian space. Throughout the paper, \( F^n \) or \( (M^n, L) \) will denote the n-dimensional finsler space, whereas n-dimensional Riemannian space will be denoted by \( R^n \).

1.2 Intrinsic Fields of Orthonormal Frames:

Berwald theory of two-dimensional Finsler space is developed based on the intrinsic field of orthonormal frame which consists of the normalized supporting element \( l^i \) and unit vector orthonormal to \( l^i \). Following idea Moor introduced, in a three-dimensional Finsler space, the intrinsic field of orthonormal frame which consists of the normalized supporting element\( l^i \), the normalized torsion vector \( C^i \) and a unit vector orthogonal to them and developed a theory of three-dimensional Finsler spaces. Generalizing the Berwald’s and Moor’s ideas, Miron and Matsumoto (1986), (1977), (1989)] developed a theory of intrinsic orthonormal frame fields on n-dimensional Finsler space as follows.
Let $L(x, y)$ be the fundamental function of an n-dimensional Finsler space and introduce Finsler tensor fields of (0, 2α−1) type, $\alpha = 1, 2, \ldots, n$ by

$$L_{(i_1 i_2 \ldots i_{2a-1}} = \frac{1}{2a} \delta_{i_1 i_2} \ldots \delta_{i_{2a-1}} L^2$$

Then we get a sequence of covariant vectors

$$L_{(i_j i_{j+1}} = L_{i_j i_{j+1} \ldots i_{2a-3}/2a-2} g^{i_1 j} g^{i_2 j} \ldots g^{i_{2a-3}/2a-2}$$

**Definition-1:** If (n-1) covariant vectors $L_{(i_j)}$, $\alpha = 1, 2, \ldots, n-1$ are linearly independent, the Finsler space is called strongly non-Riemannian.

Assuming above n-covectors $L_{(i_j)}$ are linearly independent and put $e^{i_1 j} = L_{i_1 j}/L = l^{i_1}$. Here and in following we use raising and lowering of indices as $L_{i_1 j} = g^{i_1 j} L_{i_1 j}$. Further putting $N_{ij} = g_{ij} - e_{ij} e^{ij}$ and matrix $N_1 = N_{ij}$ is of rank (n-1). Second vector $e^{2 j}$ is introduced by $e^{2 j} = L_{2 j}/L_2$, where, $L_2$ is the length of $L_{(2 j)}$ relative to $y^i$. Next we put $N_{2 j} = N_{ij} - e_{2 j} e^{2 j}$, $E_{1 j} = N_{2 j} L_{1 j}$ and so third vector $e^{3 j}$ is defined by,

$$e^{3 j} = E_{3 j}/E_3,$$

where, $E_3$ is the length of $E_{(3 j)}$ relative to $y^i$. The repetition of above process gives a vector $e^{r+1 j}$, $r = 1, 2, \ldots, n-1$ defined by

$$e^{r+1 j} = E^{r+1 j}/E_{r+1}$$

where, $E^{r+1 j} = N_{r+1 j} L_{r+1 j}/E_{r+1}$ is the length of $E_{(r+1 j)}$ relative to $y^i$ and $N_{r+1 j} = N_{r+1 j} - e_{r+1 j} e^{r+1 j}$.

**Definition-2:** The orthonormal frame $\{e_{\alpha} \}$, $\alpha = 1, 2, \ldots, n$ as above defined in every co-ordinate neighborhood of a strongly non-Riemannian Finsler space is called the ‘Miron Frame’.

Consider the Miron frame $\{e_{\alpha} \}$, If a tensor $T_i^j$ of (1, 1)-type, for instance, is given then

$$T_i^j = T_{a b} e_{a \alpha} e_{b \beta}$$

where, the scalars $T_{a b}$ are defined as

$$T_{a b} = T_j^i e_{a \alpha} e_{b \beta}.$$

These scalars $T_{a b}$ are called the scalar components of $T_i^j$ with respect to Miron frame.

Let $H_{a b \gamma}$ be scalar components of the h-covariant derivatives $e_{(a \gamma j)}$ and $V_{a b \gamma}$ be scalar components of the v-covariant derivatives $e_{(a \beta j)}$ with respect to $\Gamma$ of the vector $e_{a \gamma}$ belonging to the Miron frame. Then

$$e_{(a \gamma j)} = H_{a b \gamma} e_{b \beta} e_{(j \gamma)},$$

$$e_{(a \beta j)} = V_{a b \gamma} e_{b \beta} e_{(j \gamma)},$$

where, the scalars $H_{a b \gamma}$ and $V_{a b \gamma}$ satisfying the following relations [Berwald (1947)].

$$H_{a b \gamma} = 0, H_{a b \gamma} = -H_{b a \gamma},$$

$$V_{a b \gamma} = \delta_{a b} - \delta_{a \beta} \delta_{b j}$$

**Definition-3:** The scalars $H_{a b \gamma}$ and $V_{a b \gamma}$ are called connection scalars.

If $C_{a b \gamma}$ be the scalar components of the (h)hv-torsion tensor $C_{\gamma k}^j$ i.e.,

$$LC_{\gamma k} = C_{a b \gamma} e_{(a \gamma j)} e_{b \beta} e_{(j \gamma)}$$

then [Hambo, H(1934)], we have

**Proposition-1:**

i. $C_{1 b \gamma} = 0$

ii. $C_{2 a b} = L_{a \gamma}, C_{3 a b} = \ldots \ldots = C_{n a b} = 0$ for $n \geq 3$, where $C_{i}$ is the length of $C_{\gamma}$.

Now, we consider scalar components of covariant derivatives of a tensor field, for instance, $T_i^j$. Let $T_{a b \gamma}$ and $T_{a b \gamma}$ be the scalar components of h-and v-covariant derivatives with respect to $\Gamma$ respectively of a tensor $T_i^j$ i.e.,

$$T_i^j|_{k} = T_{a b \gamma} e_{a \alpha} e_{b \beta} e_{(j \gamma)} k$$

and

$$LT_i^j|_{k} = T_{a b \gamma} e_{a \alpha} e_{b \beta} e_{(j \gamma)} k$$

then we have [Hambo, H(1934)],

$$T_{a b \gamma} = T_{a b \gamma} e_{a \alpha} e_{b \beta} e_{(j \gamma)}$$

$$T_{a b \gamma} = \frac{1}{2a} \delta_{a b} T_{a b \gamma} + T_{a b \gamma} H_{a \gamma} + T_{a b \gamma} V_{a b \gamma}$$

$$LT_{a b \gamma} = \frac{1}{2a} \delta_{a b} T_{a b \gamma} + T_{a b \gamma} H_{a \gamma} + T_{a b \gamma} V_{a b \gamma}$$

The scalar components $T_{a b \gamma}$ and $T_{a b \gamma}$ are called h-and v-scalar derivative of $T_{a b \gamma}$ respectively.
(i) Two-dimensional Finsler space

The Miron frame \( \{e_{1j}, e_{2j}\} \) is called the Berwald frame. The first vector \( e^1_{ij} \) is the normalized supporting element \( l^i = y^i/L \) and the second vector \( e^2_{ij} = m^i \) is the unit vector orthogonal to \( l^i \). If \( C^i \) has non-zero length \( C^i \) then \( C^i = \pm C^i/C \). The connection scalars \( H_{\alpha\beta\gamma} \) and \( V_{\alpha\beta\gamma} \) of a two-dimensional Finsler space are such that

\[
H_{\alpha\beta\gamma} = 0, \quad V_{\alpha\beta\gamma} = \delta_{\alpha\beta}^{\gamma}, \text{which implies}
\]

\[
l^i_{ij} = 0, \quad m^i_{ij} = 0, \quad L^i l^j = m^i m_j, \quad L^i m^j = -l^i m_j (1.2.5)
\]

There is only one surviving scalar component of \( LC_{ijk} \) namely \( C_{222} \). If we put \( C = C_{222} \).

*Proposition-2:* In a two-dimensional Finsler space

i. The \( h \)-curvature tensor \( R_{hijk} \) of CT is written as,

\[
R_{hijk} = R(l^i m_i - l^i m_k)(l^j m_k - l^j m_i)
\]

ii. The \( h \)-torsion tensor \( P_{hijk} \) of CT is written as,

\[
P_{hijk} = l^i (l^j m_i - l^j m_k) m^j m_k
\]

iii. The \( (v)h \)-curvature tensor \( P_{ijk} \) is written as,

\[
P_{ijk} = l^i m^j m^k
\]

(ii) Three-dimensional Finsler space

The Miron frame of a three-dimensional杨䒯空间 is called the Moor-frame. The first vector \( e^1_i \) of Moor-frame \( \{e_1, e_2, e_3\} \) is the normalized supporting element \( l^i \), the second vector \( e^2_i \) is the normalized torsion vector \( m^i \) and the third \( e^3_i \) is constructed by,

\[
n^i = e^{ijk} e^1_j e^{2} e^3_k \text{where} \quad e^{ijk} = g^{(1/2)} \delta_{123}^{ijk}
\]

Now, following two Finsler vector fields are defined [Matsushima(1986)]

\[
h_i = h_i e^1_i \text{and} \quad v_i = v_i e^1_i \text{then we have},
\]

\[
H_{\alpha\beta\gamma} = \begin{bmatrix}
0 & \delta_1^\alpha & \delta_1^\beta \\
0 & 0 & h_\gamma \\
0 & -h_\gamma & 0
\end{bmatrix}, \quad V_{\alpha\beta\gamma} = \begin{bmatrix}
0 & \delta_2^\gamma & \delta_3^\gamma \\
\delta_2^\gamma & 0 & v_\gamma \\
-\delta_3^\gamma & -v_\gamma & 0
\end{bmatrix}
\]

\[
l^i_{ij} = 0, \quad L^i l^j = h^i_j
\]

\[
m^i_{ij} = n^i h_i l^j, \quad L^i m^j = -l^i m^j + n^i v^j
\]

\[
n^i_{ij} = -m^i h_i L^j m^j = -l^i n^j - m^i v^j
\]

*Definition-4:* The Finsler vector fields \( h_i \) and \( v_i \) defined in (2.2.6) are called the \( h \)-and \( v \)-connection vectors of a three-dimensional Finsler space.

The \( (h)v \)-torsion tensor of a three-dimensional Finsler space is given by [Matsushima(1986)],

\[
LC_{ijk} = H_m m_j m_k - J_n (n_j m_k) + J_m (m_k n_j) + J_n (n_j n_k)
\]

(1.2.8)

The three scalar fields \( H, l \) and \( J \) of (1.2.8) are called the main scalars of a three-dimensional Finsler space and \( \pi_{ijk} \) represent cyclic sum of the terms obtained by cyclic permutation of \( i, j, k \).

The \( h \)-and \( v \)-connection vectors of a three-dimensional space has been firstly solved, in terms of main scalars explicitly, by Ikeda (1994).

(iii) Four-dimensional Finsler space

Prof. T. N. Pandey and D. K. Dwevidi developed the theory of four-dimensional Finsler spaces in the year 1997 in terms of scalars, taking \( l^1, m^1, n^1 \) and a unit vector \( p^i \) perpendicular to \( l^1, m^1, n^1 \) as \( p^i = e^{ijk} l^1 m_1 m_j \). The orthonormal frame \( \{l^1, m^1, n^1, p^1\} \) as above defined in every coordinate neighborhood of a strongly non-Riemannian Finsler space is called Miron frame.

M. Matsumoto defines the scalar component of a tensor in Miron frame as follows:-

If a tensor \( T^i_{jk} \) of \( (1, 2) \) type for instance is given, we define scalars

\[
T_{\alpha\beta\gamma} = T^i_{jk} e_{\alpha\beta\gamma} e^i_{jk}
\]

Then \( T_{\alpha\beta\gamma} \) is written in the form,

\[
T^i_{jk} = T_{\alpha\beta\gamma} e^i_{\alpha\beta\gamma} e^j_{\alpha\beta\gamma} e^k_{\alpha\beta\gamma}
\]
These $T_{a|b}$ are called scalar components of $T_{i|k}^a$ with respect to Miron frame
\[ e_{1j}^i = l^i, \quad e_{2j}^i = m^i, \quad e_{3j}^i = n^i, \quad e_{4j}^i = p^i. \]
From the equations
\[ g_{ii} |^1 = g_{ii} m |^m = g_{ii} n |^n = g_{ii} p |^p = 1 \]
and
\[ g_{ii} |^1 m = g_{ii} |^1 n = g_{ii} |^1 p = g_{ii} m |^m = g_{ii} n |^n = g_{ii} p |^p = 0 \]
we have,
\[ g_{ij} = l_{ij} + m_{ij} + n_{ij} + p_{ij} \]
Next, the C-tensor $C_{ijk} = \frac{1}{2} \delta_{ik} b_{ij}$ satisfies $C_{ijk} = C_{ijk} = C_{ijk} = 0$. So we have the expression of $C_{ijk}$ in the form
\[ L_{ijk} = H m_i m_j + l_{ij} n_{ij} p_{ij} + 1 \pi_{ijk}(l_{ij} m_{ij} m_{ij}) \]
\[ + 1 \pi_{ijk}(l_{ij} n_{ij} n_{ij}) + 1 \pi_{ijk}(l_{ij} p_{ij} p_{ij}) \]
where, $H, I, J, K, L, J, L$ are called main scalars satisfying $H + I + K = LC$.

Now we denote the $h$- and $v$-covariant differentiations of a tensor field with respect to $C^i$ by the short line $|i)$ and long line $\iota(i)$ respectively, the following equations are derived
\[ \begin{aligned}
L_{ij} &= 0, & l_{ij} &= 0, & m_{ij} &= n_{ij} - p_{ij}, & n_{ij} &= p_{ij} = m_{ij} = n_{ij}, & l_{ij} &= h_{ij}, & L_{n_{ij}} &= H_{ij} + n_{ij} = n_{ij} + p_{ij}, & L_{p_{ij}} &= p_{ij} = n_{ij} - m_{ij}, & L_{w_{ij}} &= p_{ij} = m_{ij} - n_{ij} - w_{ij}
\end{aligned} \]

The surviving scalar components of $h_{ij} |^l$ and $v_{ij} |^l$ are given by
\[ \begin{aligned}
V_{13} &= V_{23} = V_{33} = V_{43} = 0, & V_{21} &= V_{12} = -V_{31} = V_{21} = -V_{21} = V_{32} = V_{42} = V_{42} = 0, & H_{33} &= H_{32} = H_{32} = H_{32} = 0, & H_{43} &= H_{43} = H_{43} = H_{43} = 0,
\end{aligned} \]
where $h_{ij}, v_{ij}, k_{ij}$ and $(u_{ij}, v_{ij}, w_{ij})$ are scalar components of $o$, and $i$ respectively $h_{ij} = h_{ij} e_{ij}, v_{ij} = j e_{ij}, k_{ij} = k_{ij} e_{ij}, u_{ij} = u_{ij} e_{ij}, v_{ij} = v_{ij} e_{ij}, w_{ij} = w_{ij} e_{ij}$. The first scalar component $v_{ij}$ in $l_{ij} |^l$ vanishes identically in a four-dimensional Finsler space.

The $h$-scalar derivative of the reduced components $T_{αβ}^{α'}$ of the tensor $T_{ij}^{α}$ of (1, 1) type is defined as
\[ T_{αβ}^{α'} = \frac{δT_{αβ}}{δx^α} e^{β'}_γ + T_{αβ}^{α} H_{γ|β} + T_{αβ} H_{γβ} \]
where $δ_{αβ} = \frac{δ}{δx^α} - G^{β}_{γ} \frac{δ}{δy^γ}$ and $G^{α}_{β}$ are non-linear connection of $CT^\gamma$. Similarly the $v$-scalar derivative of the adapted components $T_{αβ}^{α'}$ of $T$ is defined as,
\[ T_{αβ}^{α'} = L \frac{δT_{αβ}}{δy^α} e^{β'}_β + T_{αβ}^{α} V_{β|γ} + T_{αβ} V_{γβ} \]
Thus $T_{αβ}^{α'}$ and $T_{αβ}^{α'}$ are adopted components of $T_{i|k}$ and $LT_{i|k}$ respectively. i.e
\[ T_{i|k} = T_{αβ}^{α'} e^{α'}_α e^{β'}_β e^{γ'}_γ \]
\[ LT_{i|k} = T_{αβ}^{α'} e^{α'}_α e^{β'}_β e^{γ'}_γ \]

II. Geodesics

The curve for shortest length, measured along the surface between any two points on the surface is called geodesic curve or geodesic.

Geodesic Equation from Geodesic Curve with Finsler Space
We know that
\[ ds^2 = \sqrt{g_{αβ} \frac{dx^α}{ds} \frac{dx^β}{ds}} ds \]
\[ \Rightarrow \int ds = \int \left( -g_{a\beta} \frac{dx^a}{ds} \frac{dx^\beta}{ds} \right)^{1/2} ds \]

To extremize length take \( ds = 0 \)

According to the Euler–Lagrange equation

\[ \frac{d}{ds} \left( g_{a\beta} \frac{dx^a}{ds} \right) = \frac{1}{2} g_{a\beta} u^a u^\beta \]

Where \( -g_{a\beta} \frac{dx^a}{ds} \frac{dx^\beta}{ds} = 1 \) and \( u^a = \frac{dx^a}{ds} \)

Hence \( \frac{d}{ds} \left( g_{a\beta} u^a \right) = g_{a\beta} \frac{du^a}{ds} + g_{a\alpha} \frac{du^\alpha}{ds} u^\beta \)

Hence equation (2.1) becomes

\[ g_{a\beta} \frac{d^2 x^a}{dx^2} \frac{dx^\beta}{ds} = u^a u^\beta \left( g_{a\beta} + \frac{1}{2} g_{a\gamma} \frac{dx^a}{ds} \frac{dx^\gamma}{ds} \right) = 0 \]

Now we use

\[ u^a u^\beta g_{a\beta} \frac{dx^a}{ds} \frac{dx^\beta}{ds} = u^a u^\beta \frac{1}{2} \left( g_{a\beta} + g_{a\gamma} \frac{dx^a}{ds} \frac{dx^\gamma}{ds} \right) \]

And multiply equation (2.3) by \( g^{\beta\gamma} \) to obtain

\[ \frac{d^2 x^a}{ds^2} + \frac{1}{2} g^{\beta\gamma} \left( g_{a\beta} + g_{a\gamma} \frac{dx^a}{ds} \frac{dx^\gamma}{ds} - g_{a\gamma} \frac{dx^a}{ds} \frac{dx^\gamma}{ds} \right) = 0 \]

Since a co-ordinate form

\[ r^\alpha_{vi} = g_{a\beta} \frac{dx^a}{ds} \]

Again from (1.1.1) & (1.1.9) We have

\[ ds = L(x^i, dx^j) = [g_{ij}(x^k) dx^i dx^j]^{1/2} \]

From geodesics integral

\[ g_{a\beta} = -g_{ij}(x^k) \frac{dx^i}{dx^a} \frac{dx^j}{dx^\beta} \]

Again from (2.2) we have

\[ g_{a\beta} \frac{dx^a}{ds} \frac{dx^\beta}{ds} = \frac{d}{ds} \left( g_{a\beta} u^a \right) - g_{a\beta} \frac{du^a}{ds} \frac{du^\beta}{ds} \]

\[ g_{a\beta} \frac{du^a}{ds} \frac{du^\beta}{ds} = \frac{d}{ds} \left( g_{a\beta} u^a \right) - g_{a\beta} \frac{du^a}{ds} \frac{du^\beta}{ds} \]

\[ g_{a\beta} \frac{du^a}{ds} \frac{du^\beta}{ds} = \frac{d}{ds} \left( g_{a\beta} u^a \right) - g_{a\beta} \frac{du^a}{ds} \frac{du^\beta}{ds} \]

From (2.6), (2.7), (2.8) & (2.9) we have the geodesics equation of the form

\[ \frac{d^2 x^a}{ds^2} + g^{\beta\gamma} \left[ g_{ij}(x^k) \frac{dx^i}{dx^\alpha} \frac{dx^j}{dx^\gamma} - g_{ij}(x^k) \frac{dx^i}{dx^\alpha} \frac{dx^j}{dx^\gamma} - g_{ij}(x^k) \frac{dx^i}{dx^\alpha} \frac{dx^j}{dx^\gamma} - g_{ij}(x^k) \frac{dx^i}{dx^\alpha} \frac{dx^j}{dx^\gamma} \right] = 0 \]

**Theorem**

The geodesic of the velocity space metric defined in \( ds^2 = dx^2 + \sin \theta^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \),

where the magnitude of the velocity is \( v = \tan n \theta \) are paths of minimum fuel for a rocket ship changing its velocity.

**Proof**—the geodesic is the path between two velocities which minimizes the arc–length between them, but arc–length in the velocity space is just the magnitude of a small change of velocity. Science a rocket expends fuel monotonically for the boost it requires the geodesic of velocity space are paths of minimum fuel use.

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