Fixed Point Theorem For \( \emptyset \) - Weakly Expansive Mappings And R-Weakly Commuting Mappings In Metric Spaces

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Abstract: In this paper, we prove common fixed point theorems for \( \emptyset \)-weakly expansive mappings, which generalize and extend the results of S. M. Kang\([10]\) using the concept of weak reciprocal continuity in metric spaces. We introduce the concept of \( \emptyset \)-weakly expansive mappings.

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I. Introduction

In 1997, Alber and Guerre-Delabriere \([11]\) introduced the notion of \( \emptyset \)-weakly contractive. We introduce the notion of \( \emptyset \)-weakly expansive mappings in metric space. In 1986, Jungck \([2]\) introduced the notion of compatible mappings. In 1994, Pant \([4]\) introduced the notion of R-weak commutativity in metric spaces to extend the scope of the study of common fixed point theorems from the class of weakly commuting mappings to wider class of R-weakly commuting mappings. In 1997, Pathak et al. \([3]\) improved the notion of R-weakly commuting mappings to R-weakly commutating mappings of type \((A_f)\) and of type \((A_g)\). In 1998 and 1999, Pant \([5, 6]\) introduced a new notion of continuity, known as reciprocal continuity. Recently, Pant et al. \([7]\) generalized the notion of reciprocal continuity to weak reciprocal continuity. In 2012, Manro and Kumar \([9]\) proved the following fixed point theorem in complete metric spaces: In 1922, Banach proved a common fixed point theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. This result of Banach is known as Banachs fixed point theorem or Banach contraction principle.

II. Preliminaries

Definition: Let \( F \) be a self mapping of a metric space \((X, d)\). Then \( F \) is said to be expansive if there exists a real number \( h > 1 \) such that \( d(Fx, Fy) \geq hd(x, y) \) for all \( x, y \in X \).

Definition: Let \( F \) be a self mapping of a metric space \((X, d)\). Then \( F \) is said to be \( \phi \)-weakly contractive if there exists a continuous mapping \( \emptyset : [0, \infty) \rightarrow [0, \infty) \) with \( \emptyset(0) = 0 \) and \( \emptyset(t) < t \) for all \( t > 0 \) such that
\[
d(Fx, Fy) \leq \emptyset(d(x, y)),
\]
for all \( x, y \in X \).

Definition: Let \( F \) be a self mapping of a metric space \((X, d)\). Then \( F \) is said to be \( \phi \)-weakly expansive if there exists a continuous mapping \( \emptyset : [0, \infty) \rightarrow [0, \infty) \) with \( \emptyset(0) = 0 \) and \( \emptyset(t) > t \) for all \( t > 0 \) such that
\[
d(Fx, Fy) \geq \emptyset(d(x, y)),
\]
for all \( x, y \in X \).

Definition: Let \( F \) and \( G \) be two self mappings of a metric space \((X, d)\). Then \( F \) is said to be \( \phi \)-weakly expansive with respect to \( G \) : \( X \rightarrow X \) if there exists a continuous mapping \( \emptyset : [0, \infty) \rightarrow [0, \infty) \) with \( \emptyset(0) = 0 \) and \( \emptyset(t) > t \) for all \( t > 0 \) such that
\[
d(Fx, Fy) \geq \emptyset(d(Gx, Gy)),
\]
for all \( x, y \in X \).

Definition: Let \( F \) and \( G \) be two self mappings of a metric space \((X, d)\). Then \( F \) is said to be compatible if \( d(FGx_n, GFx_n) = 0 \), whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} Fx_n = \lim_{n \to \infty} Gx_n = t \) for some \( t \in X \). An immediate consequence is that if \( F \) and \( G \) are compatible and \( Fz = Gz \), \( z \) is called a coincidence point of \( F \) and \( G \), then \( FGz = GFz \).

Definition: Let \( F \) and \( G \) be two self mapping of a metric space \((X, d)\). Then \( F \) and \( G \) are said to be R-weakly commuting if there exists \( R > 0 \) such that \( d(FGx, GFx) \leq Rd(Fx, Gx) \) for all \( x \in X \).

Definition: Let \( F \) and \( G \) be two self mapping of a metric space \((X, d)\). Then \( F \) and \( G \) are said to be 1. R-weakly commuting of type \((A_0)\) if there exists \( R > 0 \) such that \( d(FFx, GFx) \leq Rd(Fx, Gx) \) for all \( x \in X \).
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1. R-weakly commuting of type (A_p) if there exists some R > 0 such that 
\[ d(FGx, GGx) \leq Rd(Fx, Gx) \]
for all \( x \in X \).

**Definition:** Let F and G be two self mapping of a metric space \((X, d)\). Then F and G are said to be R-weakly commuting of type (P) if there exists R > 0 such that 
\[ d(FFx, GGx) \leq Rd(Fx, Gx) \]
for all \( x \in X \).

**Definition:** Let F and G be two self mappings of a metric space \((X, d)\). Then F and G are said to be reciprocally continuous if 
\[ \lim_{n \to \infty} FGx_n = Ft \quad \text{and} \quad \lim_{n \to \infty} GFx_n = Gt \]
whenever \( \{x_n\} \) is a sequence in \( X \) such that 
\[ \lim_{n \to \infty} Fx_n = \lim_{n \to \infty} Gx_n = t \]
for some \( t \in X \).

If F and G are both continuous, then they are obviously continuous, but the converse need not be true.

**Definition:** Let F and G be two self mappings of a metric space \((X, d)\). Then F and G are said to be weakly reciprocally continuous if 
\[ \lim_{n \to \infty} FGx_n = Ft \quad \text{or} \quad \lim_{n \to \infty} GFx_n = Gt \]
whenever \( \{x_n\} \) is a sequence in \( X \) such that 
\[ \lim_{n \to \infty} Fx_n = \lim_{n \to \infty} Gx_n = t \]
for some \( t \in X \).

If F and G are both reciprocally continuous, then they are obviously weakly reciprocally continuous, but the converse need not be true.

**III. Main Result**

**Fixed Point Theorem For \( \emptyset \) - Weakly Expansive Mapping**

**Theorem 3.1:** Let M and D be two weakly reciprocally continuous self mappings of a complete metric space \((X, d)\) satisfying

1. \( D(X) \subset M(X) \);
2. There exists a continuous mapping \( \emptyset : [0, \infty) \to [0, \infty) \) with \( \emptyset(0) = 0 \) and \( \emptyset (t) > t \) for all \( t > 0 \) such that 
\[ d(Mx, My) \geq N(Dx, Dy) + \emptyset(N(Dx, Dy)) \]

Where,
\[ N(Dx, Dy) = \min\{d(Dx, Dy), d(Mx, Dx), d(My, Dy), d(Mx, My)\} \]

For all \( x, y \in X \).

If M and D are compatible, then M and D have a unique common fixed point in X.

**Proof:** Let \( x_0 \) be any point in X. Since \( D(X) \subset M(X) \), there exists a sequence \( \{x_n\} \) such that 
\[ Dx_n = Mx_{n+1} \]
Define a sequence \( \{y_n\} \) in X by
\[ y_{n+1} = Dx_n = Mx_{n+1} \]

**Case 1:** We assume that if \( y_n = y_{n+1} \) for some \( n \in N \), there is nothing to prove.

**Case 2:** We assume that \( y_n \neq y_{n+1} \) for all \( n \in N \), we have 
\[ d(y_n, y_{n-1}) = d(Mx_{n+1}, My_n) \]
\[ \geq \min\{d(Dx_{n+1}, Dx_n), d(Mx_{n+1}, Dx_n), d(Mx_n, Dx_n), d(Mx_{n+1}, d(Mx_n, Dx_n))\} + \emptyset\{\min\{d(Dx_{n+1}, Dx_n), d(Mx_{n+1}, Dx_n), d(Mx_n, Dx_n), d(Mx_{n+1}, d(Mx_n, Dx_n))\}\} \]
\[ \geq \min\{d(y_{n+1}, y_{n+2}), d(y_{n+1}, y_{n+2}), d(y_{n+1}, y_{n+2}), d(y_{n+1}, y_{n+2})\} + \emptyset\{\min\{d(y_{n+1}, y_{n+2}), d(y_{n+1}, y_{n+2}), d(y_{n+1}, y_{n+2}), d(y_{n+1}, y_{n+2})\}\} \]
\[ \geq d(y_{n+1}, y_{n}) + \emptyset(d(y_{n+1}, y_{n})) \]

That is,
\[ d(y_n, y_{n-1}) \geq d(y_{n+1}, y_{n}) \]

Hence the sequence \( \{d(y_{n+1}, y_{n})\} \) is strictly decreasing and bounded below. Thus there exists \( r \geq 0 \) such that 
\[ \lim_{n \to \infty} d(y_{n+1}, y_{n}) = r \]
Letting \( n \to \infty \) in (3.2) we get \( r \geq \emptyset (r) \), which is a contradiction. Hence we have \( r = 0 \). Therefore 
\[ \lim_{n \to \infty} d(y_{n+1}, y_{n}) = 0 \]

Now we will show that \( y_n \) is a Cauchy sequence.

Let \( y_n \) is not a Cauchy sequence. Then there exists an \( \varepsilon > 0 \) and the subsequence \( \{y_{m(k)}\} \) and \( \{y_{n(k)}\} \) of \( y_n \) such that minimal \( n(k) \) in the sense that \( n(k) > m(k) > k \) and 
\[ d(y_{m(k)}, y_{n(k)}) > \varepsilon \]. Therefore 
\[ d(y_{m(k)}, y_{n(k)-1}) \geq \varepsilon \].

By the triangular inequality, we have 
\[ \varepsilon < d(y_{m(k)}, y_{n(k)}) \]
\[ \leq d(y_{m(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)}) \]
\[ \leq d(y_{m(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)}) + d(y_{m(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)}) \]
\[ \leq d(y_{m(k)}, y_{m(k)-1}) + \varepsilon + d(y_{n(k)-1}, y_{n(k)}) \]
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Letting \( k \to \infty \) in the above inequality and using (3.3) we get,
\[
\lim_{k \to \infty} d(y_{m(k)}, y_{n(k)}) = \lim_{k \to \infty} d(y_{m(k)-1}, y_{n(k)-1}) = \varepsilon
\]
(3.4)

From (2), we have
\[
d(y_{m(k)-1}, y_{n(k)-1}) = d(Mx_{m(k)}, Mx_{n(k)})
\]
\[
\geq \min \{ d(Dx_{m(k)}, Dx_{m(k)}), d(Mx_{m(k)}, Dx_{m(k)}), d(Mx_{m(k)}, Mx_{m(k)}), d(Mx_{m(k)}, Dn_{m(k)}) \}
+ \emptyset \left[ \min \{ d(Dx_{n(k)}, Dx_{n(k)}), d(Mx_{n(k)}, Dx_{n(k)}), d(Mx_{n(k)}, Mx_{n(k)}), d(Mx_{n(k)}, Dn_{n(k)}) \} \right]
\]
\[
\geq \min \{ d(y_{m(k)+1}, y_{n(k)+1}) + d(y_{m(k)}, y_{n(k)}), d(y_{m(k)}, y_{n(k)+1}), d(y_{m(k)}, y_{n(k)}) \}
+ \emptyset [d(y_{m(k)}, y_{n(k)})]
\]
\[
\geq d(y_{m(k)}, y_{n(k)}) + \emptyset [d(y_{m(k)}, y_{n(k)})]
\]

Letting \( k \to \infty \), and using (3.4) we get \( \varepsilon \geq \varepsilon + \emptyset (\varepsilon) \), which is contradiction, since \( \emptyset (\varepsilon) > \varepsilon \). Hence \( \{y_n\} \) is not a Cauchy sequence in \( X \). Since \( X \) is complete there exists a point \( z \in X \) such that \( \lim_{n \to \infty} y_n = z \). Therefore by (3.1) we have
\[
\lim_{n \to \infty} y_{n+1} = \lim_{n \to \infty} Dx_n = \lim_{n \to \infty} Mx_{n+1} = z
\]

Suppose that \( M \) and \( D \) are compatible mappings. Now, by weak reciprocal continuity of \( M \) and \( D \), we obtain
\[
\lim_{n \to \infty} MDx_n = Mz \text{ or } \lim_{n \to \infty} DMx_n = Dz.
\]

Let \( \lim_{n \to \infty} MDx_n = Mz \). Then the compatibility of \( M \) and \( D \) gives
\[
\lim_{n \to \infty} d(MDx_n, DMx_n) = 0
\]

Hence,
\[
\lim_{n \to \infty} DMx_n = Mz
\]

Now we claim that \( Mz = Dz \). Let \( Mz \neq Dz \). From (3.1), we get
\[
\lim_{n \to \infty} MDx_n + 1 = \lim_{n \to \infty} DMx_n = Mz.
\]

Therefore from (2), we get
\[
d(Mz, MDx_n) \geq \min \{ d(Dz, MDx_n), d(Mz, Dz), d(MDx_n, Dz), d(Dz, MDx_n), d(Mz, Dz) \}
+ \emptyset [d(Dz, MDx_n), d(Mz, Dz), d(MDx_n, Dz), d(Dz, MDx_n), d(Mz, Dz)]
\]

Letting \( n \to \infty \), we get
\[
\geq \min \{ d(Dz, Mz), d(Mz, Dz), d(Mz, Mz), d(Mz, Dz) \}
+ \emptyset [d(Dz, Mz), d(Mz, Dz), d(Mz, Mz), d(Mz, Dz)]
\]
\[
\geq d(Mz, Dz) + \emptyset [d(Mz, Dz)]
\]
\[
> 2 d(Mz, Dz)
\]

Which is a contradiction. Hence \( Mz = Dz \). Again the compatibility of \( M \) and \( D \) implies that commutativity at a coincidence point. Hence \( DMz = MDz = MMz = DDz \).

Using (2), we obtain
\[
d(Dz, DDz) = d(Mz, MDz)
\]
\[
\geq \min \{ d(Dz, DDz), d(Mz, Mz), d(Mz, Mz), d(Mz, Mz) \}
+ \emptyset [d(Dz, Mz), d(Dz, Mz), d(Mz, Mz), d(Mz, Mz)]
\]
\[
\geq \min \{ d(Dz, DDz), d(Dz, DDz), d(Dz, DDz), d(Dz, DDz) \}
+ \emptyset [d(Dz, DDz), d(Dz, DDz), d(Dz, DDz), d(Dz, DDz)]
\]
\[
\geq d(Dz, DDz) + \emptyset [d(Dz, DDz)]
\]

Which implies that \( Dz = DDz \). Also we get \( Dz = DDz = MDz \) and so \( Dz \) is a common fixed point of \( M \) and \( D \).

Next, suppose that \( \lim_{n \to \infty} MDx_n = Dz \). Since \( D(X) \subseteq M(X) \) there exists \( u \in X \) such that \( Dz = Mu \) and therefore \( \lim_{n \to \infty} MDx_n = Mu \). The compatibility of \( M \) and \( D \) implies that \( \lim_{n \to \infty} MDx_n = Mu \). Now, we prove that \( Mu = Du \). Let \( Mu \neq Du \). By (3.1), we have
\[
\lim_{n \to \infty} DM_{x_{n+1}} = \lim_{n \to \infty} DD_{x_n} = Mu
\]

From (2), we have
\[
d(Mu, MD_{x_n}) \geq \min\{d(Du, DD_{x_n}), d(Mu, Du), d(MD_{x_n}, DD_{x_n}), d(Mu, MD_{x_n}), d(Mu, DD_{x_n})\}
+ \emptyset[\min\{d(Du, DD_{x_n}), d(Mu, Du), d(MD_{x_n}, DD_{x_n}), d(Mu, MD_{x_n}), d(Mu, DD_{x_n})\}]
\]

Letting \( n \to \infty \), we get
\[
d(Mu, Mu) \geq \min\{d(Du, Mu), d(Mu, Du), d(Mu, Mu), d(Mu, Mu), d(Mu, Mu)\} + \emptyset[\min\{d(Du, Mu), d(Mu, Du), d(Mu, Mu), d(Mu, Mu), d(Mu, Mu)\}]
\]
\[
\geq d(Mu, Du) + \emptyset[d(Mu, Du)]
\]
\[
> 2d(Mu, Du)
\]

Which is a contradiction. Hence \( Mu = Du \). Again the compatibility of \( M \) and \( D \) implies that commutativity at a coincidence point. Hence \( DMu = MDu = MMu = DDu \). Finally Using (2), we obtain
\[
d(Du, DDu) = d(Mu, MDu)
\]
\[
\geq \min\{d(Du, Du), d(Mu, Du), d(MDu, DDu), d(Mu, MDu), d(Mu, DDu)\} + \emptyset[\min\{d(Du, Du), d(Mu, Du), d(MDu, DDu), d(Mu, MDu), d(Mu, DDu)\}]
\]
\[
\geq \min\{d(Du, Du), d(Du, Du), d(DDu, DDu), d(Du, DDu), d(Du, DDu)\} + \emptyset[\min\{d(Du, Du), d(Du, Du), d(DDu, DDu), d(Du, DDu), d(Du, DDu)\}]
\]
\[
\geq d(Du, DDu) + \emptyset[d(Du, Du)]
\]

Which implies that \( Du = DDu \). Also we get \( Du = DDu = MMu = DDu \) and so Du is a common fixed point of \( M \) and \( D \).

**Uniqueness:** Let \( v \) and \( w(v \neq w) \) be two common fixed point \( M \) and \( D \). From (2), we have
\[
d(v, w) = d(Mv, Mw)
\]
\[
\geq \min\{d(Dv, Dw), d(Mv, Dv), d(Mw, Dw), d(Mv, Mw), d(Mv, Dw)\} + \emptyset[\min\{d(Dv, Dw), d(Mv, Dv), d(Mw, Dw), d(Mv, Mw), d(Mv, Dw)\}]
\]
\[
\geq \min\{d(v, w), d(v, v), d(w, w), d(v, v), d(v, v), d(v, v)\} + \emptyset[\min\{d(v, w), d(v, v), d(w, w), d(v, v), d(v, v)\}]
\]
\[
\geq d(v, w) + \emptyset[d(v, w)]
\]

Which implies that \( v = w \). Hence \( M \) and \( D \) have a unique common fixed point.

**Fixed Point Theorem For R-Weakly Commuting of Type \((A_g)\) and Type \((A_f)\)**

**Theorem 3.2:** Let \( M \) and \( D \) be two weakly reciprocally continuous self mappings of a complete metric space \((X, d)\) satisfying
1. \( D(X) \subseteq \text{M}(X) \);
2. There exists a continuous mapping \( \emptyset : [0, \infty) \to [0, \infty) \) with \( \emptyset (0) = 0 \) and \( \emptyset (t) > t \) for all \( t > 0 \) such that
\[
d(Mx, My) \geq N(Dx, Dy) + \emptyset(N(Dx, Dy))
\]
Where,
\[
N(Dx, Dy) = \min\{d(Dx, Dy), d(Mx, Dx), d(My, Dy), d(Mx, My), d(Mx, Dy)\}
\]
For all \( x, y \in X \). If \( M \) and \( D \) are R-weakly commuting of type \((A_g)\) and type \((A_f)\), then \( M \) and \( D \) have a unique common fixed point in \( X \).

**Proof:** From above theorem \( \{y_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete there exists a point \( z \in X \) such that \( \lim_{n \to \infty} y_n = z \). Therefore by (3.1) we have
\[
\lim_{n \to \infty} y_{n+1} = \lim_{n \to \infty} D_{x_n} = \lim_{n \to \infty} M_{x_{n+1}} = z
\]

Now, suppose that \( M \) and \( D \) are R-weakly commuting of type \((A_f)\) . The weak reciprocal continuity of \( M \) and \( D \) implies that \( \lim_{n \to \infty} MD_{x_n} = Mz \) or \( \lim_{n \to \infty} DM_{x_n} = Dz \).
Let \( \lim_{n \to \infty} MDx_n = Mz \). Then the R-weakly commuting of type \( (A_f) \) of M and D yields,
\[
d(DDx_n, MDx_n) \leq Rd(Mx_n, Dx_n)
\]
and therefore \( \lim_{n \to \infty} d(DDx_n, Mz) \leq Rd(z, z) = 0 \), that is
\( \lim_{n \to \infty} DDx_n = Mz \).

Now we claim that \( Mz = Dz \). Let \( Mz \neq Dz \). From (2), we get
\[
d(Mz, MDx_n) \geq \min\{d(Dz, DDx_n), d(Mz, Dz), d(MDx_n, DDx_n), d(Mz, DDx_n)\} + \varnothing[d(Dz, DDx_n), d(Mz, Dz), d(MDx_n, DDx_n), d(Mz, DDx_n)]
\]
Letting \( n \to \infty \), we get
\[
\geq \min\{d(Dz, Mz), d(Mz, Dz), d(Mz, Mz), d(Mz, Mz)\} + \varnothing[d(Mz, Dz)]
\]
Which is a contradiction. Hence \( Mz = Dz \).
Again by R-weakly commutativity of type \( (A_f) \) \( d(DDz, MDz) \leq Rd(Dz, Mz) = Rd(z, z) = 0 \) that is \( DDz = MDz \).
Therefore \( DMz = MDz = MMz = DDz \). Using (2), we obtain
\[
d(Dz, DDz) = d(Mz, MDz)
\]
\[
\geq \min\{d(Dz, DDz), d(Mz, Dz), d(MDz, DDz), d(Mz, MDz), d(Mz, DDz)\} + \varnothing[d(Dz, DDz), d(Mz, Dz), d(MDz, DDz), d(Mz, MDz), d(Mz, DDz)]
\]
\[
\geq \min\{d(Dz, DDz), d(Mz, Dz), d(MDz, DDz), d(Dz, DDz), d(Dz, DDz)\} + \varnothing[d(Dz, DDz), d(Mz, Dz), d(MDz, DDz), d(Dz, DDz)]
\]
\[
\geq d(Dz, DDz) + \varnothing[d(Dz, DDz)]
\]
Which implies that \( Dz = DDz \). Then we also get \( Dz = DDz = MDz \) and so \( Dz \) is a common fixed point of M and D. Similarly, if \( \lim_{n \to \infty} DMx_n = Dz \), we can easily prove.

Suppose that M and D are R-weakly commuting of type \( (A_f) \). Again, as done above, we can easily prove that \( Mz \) is a common fixed point of M and D.

**Uniqueness:** From theorem 3.1, we can easily prove the uniqueness of the theorem. Hence M and D have a unique common fixed point.

**Fixed Point Theorem For R-Weakly Commuting of Type (P)**

**Theorem 3.3:** Let M and D be two weakly reciprocally continuous self mappings of a complete metric space \((X, d)\) satisfying
1. \( D(X) \subset M(X) \);
2. There exists a continuous mapping \( \varnothing : [0, \infty) \to [0, \infty) \) with \( \varnothing(0) = 0 \) and \( \varnothing(t) > t \) for all \( t > 0 \) such that
\[
d(Mx, My) \geq N(Dx, Dy) + \varnothing(N(Dx, Dy))
\]
Where,
\[
N(Dx, Dy) = \min\{d(Dx, Dy), d(Mx, Dx), d(My, Dy), d(Mx, My), d(Mx, Dy), d(My, Dy)\}
\]
For all \( x, y \in X \).

If M and D are R-weakly commuting of type (P), then M and D have a unique common fixed point in X.

**Proof:** From above theorem \( \{y_n\} \) is a Cauchy sequence in X. Since X is complete there exists a point \( z \in X \) such that \( \lim_{n \to \infty} y_n = z \). Therefore by (3.1) we have
\[
\lim_{n \to \infty} y_{n+1} = \lim_{n \to \infty} Dx_n = \lim_{n \to \infty} Mx_{n+1} = z
\]

Now, suppose that M and D are R-weakly commuting of type (P). The weak reciprocal continuity of M and D, implies that \( \lim_{n \to \infty} MDx_n = Mz \) or \( \lim_{n \to \infty} MDx_n = Dz \). Let \( \lim_{n \to \infty} MDx_n = Mz \). Then the R-weakly commutativity of type (P) of M and D yields,
\[
d(MMx_n, DDx_n) \leq Rd(Mx_n, Dx_n)
\]
and therefore \( \lim_{n \to \infty} d(MMx_n, DDx_n) = Rd(z, z) = 0 \). That is \( \lim_{n \to \infty}(MMx_n, DDx_n) = 0 \). Using (3.1), we have \( MDx_{n-1} = MMx_n \to Mz \) and \( DDx_n = Mz \) as \( n \to \infty \).

Now we claim that \( Mz = Dz \). Let \( Mz \neq Dz \). From (2), we get
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$$d(Mz, MDx_n) \geq \min\{d(Dz, DDx_n), d(Mz, Dz), d(MDx_n, DDx_n), d(Mz, MDx_n), d(Mz, DDx_n)\} + \emptyset[\min\{d(Dz, Dz), d(Mz, Dz), d(Mz, MDz), d(Mz, Mz)\}]$$

Letting $n \to \infty$, we get

$$d(Mz, Dz) \geq \min\{d(Dz, Dz), d(Mz, Dz), d(MDz, Dz), d(Mz, MDz), d(Mz, Mz)\} + \emptyset[\min\{d(Dz, Dz), d(Mz, Dz), d(Mz, MDz), d(Mz, Mz)\}]$$

$$\geq d(Mz, Dz) + \emptyset[d(Mz, Dz)]$$

Which is a contradiction. Hence $Mz = Dz$. Again by using the R-weakly commutativity of type (P), we have $d(MMz, DDz) \leq Rd(Mz, Dz) = 0$ that is $DDz = MMz$.

Therefore $DDz = Mz = MDz = Dz$.

Using (2), we obtain

$$d(Dz, DDz) = d(Mz, MDz) \geq \min\{d(Dz, DDz), d(Mz, Dz), d(MDz, Dz), d(Mz, MDz), d(Mz, DDz)\} + \emptyset[\min\{d(Dz, DDz), d(Mz, Dz), d(Mz, MDz), d(Mz, DDz)\}]$$

$$\geq\min\{d(Dz, DDz), d(Dz, Dz), d(DDz, DDz), d(Dz, Dz), d(DDz, DDz)\} + \emptyset[\min\{d(Dz, DDz), d(Dz, Dz), d(DDz, DDz), d(Dz, DDz)\}]$$

$$\geq d(Dz, DDz) + \emptyset[d(Dz, DDz)]$$

Which implies that $Dz = DDz$. Therefore we also get $Dz = DDz = MDz$ and so $Dz$ is a common fixed point of $M$ and $D$. Similarly, if $\lim_{n \to \infty} Dz = Dz$, we can easily prove.

**Uniqueness:** From Theorem 3.1, we can easily prove the uniqueness of the theorem. Hence $M$ and $D$ have a unique common fixed point.

**Corollary:** Let $M$ be surjective self mappings of a complete metric space $(X, d)$ satisfying

1. there exists a continuous mapping $\emptyset : [0, \infty) \to [0, \infty)$ with $\emptyset(0) = 0$ and $\emptyset(t) > t$ for all $t > 0$ such that

$$d(Mx, My) \geq N(x, y) + \emptyset(N(x, y))$$

Where,

$$N(x, y) = \min\{d(x, y), d(Mx, x), d(My, y), d(Mx, My) d(My, y)\}$$

For all $x, y \in X$.

Then $M$ and $D$ have a unique fixed point in $X$.

**Example:** Let $X = [0, 1]$ be equipped with the Euclidean metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define $M, D : X \to X$ by $Mx = 8x$ and $Dx = 2x$, so $DX = [0, 2] \subset [0, 8] = MX$.

Let $\{x_n\}$ be a sequence in $X$ such that $x_n = \frac{1}{n}$ for each $n$. Also, let $\emptyset : [0, \infty) \to [0, \infty)$ be defined by $\emptyset(t) = 2t$ for all $t \in [0, \infty)$. Here, $Mx_n = \frac{8}{n}$ and $Dx_n = \frac{2}{n}$, so $\lim_{n \to \infty} Mx_n = 0$.

Also $\lim_{n \to \infty} MDx_n = \lim_{n \to \infty} Mx_n = 0 = M(0)$, so we can say that $M$ and $D$ are weakly reciprocally continuous. Also, $d(Mx, My) = 8|x - y|$, $d(Dx, Dy) = 2|x - y|$ and

$$\emptyset(d(Dx, Dy)) = 4|x - y|$$

Clearly,

$$d(Mx, My) = 8|x - y|$$

$$\geq 2|x - y| + \emptyset(2|x - y|)$$

$$\geq 2|x - y| + 4|x - y|$$

$$\geq 6|x - y|.$$ 

Again, $d(D Dx, Dy) = \left(\frac{2}{n}, \frac{2}{n}\right)$

$$= d(Mx_n, Mx_n) = \emptyset\left(\frac{2}{n}, \frac{2}{n}\right) = \emptyset\left(\frac{2}{n}, \frac{2}{n}\right)$$

Clearly,

$$d(Dx_n, My_n) = \emptyset\left(\frac{2}{n}, \frac{2}{n}\right) = \emptyset\left(\frac{2}{n}, \frac{2}{n}\right)$$

Hence $M$ and $D$ are R-weakly commuting mappings of type $(A_4)$, Also $M$ and $D$ are compatible. So all the conditions of Theorem 3.1 and 3.2 are satisfied and $0$ is the unique fixed point of $M$ and $D$.
IV. Conclusion

In this paper, we have presented common fixed point theorems in metric spaces through concept of $\emptyset$-weakly expansive mappings and $R$-weakly commuting mappings.

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References