Strongly $\alpha^*$ Continuous Functions in Topological Spaces

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Abstract: The Purpose Of This Paper Is To Introduce Strongly And Perfectly $\alpha^*$Continuous Maps And Basic Properties And Theorems Are Investigated. Also, We Introduced $\alpha^*$ Open And Closed Maps And Their Properties Are Discussed.

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I. Introduction

In 1960, Levine . N [3] introduced strong continuity in topological spaces. Beceren.Y [1] in 2000, introduced and studied on strongly $\alpha$ continuous functions. Also, in 1982 Malghan [5] introduced the generalized closed mappings. Recently, S. Pious Missier and P. Anbarasi Rodrigo[8] have introduced the concept of $\alpha^*$-open sets and studied their properties. In this paper we introduce and investigate a new class of functions called strongly $\alpha^*$ continuous functions. Also we studied about $\alpha^*$ open and $\alpha^*$ closed maps and their relations with various maps.

II. Preliminaries

Throughout this paper $(X, \tau)$, $(Y, \sigma)$ and $(Z, \eta)$ or $X$, $Y$, $Z$ represent non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset $A$ of a space $(X, \tau)$, cl$(A)$ and int$(A)$ denote the closure and the interior of $A$ respectively. The power set of $X$ is denoted by $P(X)$.

Definition 2.1: A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called a strongly continuous [3] if $f^{-1}(O)$ is both open and closed in $(X, \tau)$ for each subset $O$ in $(Y, \sigma)$.

Definition 2.2: A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called a $\alpha$-continuous [4] if $f^{-1}(O)$ is a $\alpha$ open set [6] of $(X, \tau)$ for every open set $O$ of $(Y, \sigma)$.

Definition 2.3: A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called a $\alpha^*$ continuous [9] if $f^{-1}(O)$ is a $\alpha^*$ open set of $(X, \tau)$ for every open set $O$ of $(Y, \sigma)$.

Definition 2.4: A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called a $g$-continuous [10] if $f^{-1}(O)$ is a $g$-open set [2] of $(X, \tau)$ for every open set $O$ of $(Y, \sigma)$.

Definition 2.5: A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called a perfectly continuous [7] if $f^{-1}(O)$ is both open and closed in $(X, \tau)$ for every open set $O$ in $(Y, \sigma)$.

Definition 2.6: A map $f: (X, \tau) \rightarrow (Y, \sigma)$ is called a $g$-closed [5] if $f(O)$ is $g$-closed in $(Y, \sigma)$ for every closed set $O$ in $(X, \tau)$.

Definition 2.7: A Topological space $X$ is said to be $\alpha^*$T$_{1/2}$ space [9] if every $\alpha^*$ open set of $X$ is open in $X$.

Theorem 2.8(8) :

(i) Every open set is $\alpha^*$ open and every closed set is $\alpha^*$-closed set
(ii) Every $\alpha$-open set is $\alpha^*$-open and every $\alpha$-closed set is $\alpha^*$-closed.
(iii) Every g-open set is $\alpha^*$-open and every g-closed set is $\alpha^*$-closed.

III. Strongly $\alpha^*$ Continuous Function

We introduce the following definition.

Definition 3.1: A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called a strongly $\alpha^*$ continuous if the inverse image of every $\alpha^*$ open set in $(Y, \sigma)$ is open in $(X, \tau)$.

Theorem 3.2: If a map $f: X \rightarrow Y$ from a topological spaces $X$ into a topological spaces $Y$ is strongly $\alpha^*$ continuous then it is continuous.
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**Proof:** Let O be a open set in Y. Since every open set is α *open, O is α *open in Y. Since f is strongly α *continuous, \( f^{-1}(O) \) is open in X. Therefore f is continuous.

**Remark 3.3:** The following example supports that the converse of the above theorem is not in general.

**Example 3.4:** Let \( X = \{ a, b, c \}, \tau = \{ \emptyset, \{a\}, \{a,b\}, \{a,b,c\} \} \) and \( \sigma = \{ \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\} \} \). Let g: \( (X, \tau) \to (Y, \sigma) \) be defined by g(a) = g(b) = a, g(c) = b. Clearly, g is not strongly α *continuous, since \( \{a\} \) is α *open set in Y but \( g^{-1}(\{a\}) = \{a,b\} \) is not an open set of X. However, g is continuous.

**Theorem 3.5:** A map f: X \to Y from a topological spaces X into a topological spaces Y is strongly α * continuous if and only if the inverse image of every α * closed set in Y is closed in X.

**Proof:** Assume that f is strongly α *continuous. Let O be any α * closed set in Y. Then \( f^{-1}(O) \) is open in X. Since f is strongly α * continuous, \( f^{-1}(O) \) is open in X. Therefore f is continuous.

**Remark 3.7:** The converse of the above theorem need not be true.

**Example 3.8:** Let X = \{ a, b, c, d \}, \tau = \{ \emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\} \} and \( \sigma = \{ \emptyset, \{a\}, \{b\}, \{a,b\} \} \). Let f: \( (X, \tau) \to (Y, \sigma) \) be defined by f(a) = f(d) = a, f(b) = b, f(c) = c. Clearly, f is strongly α *continuous. But f\(^{-1}\)(\{a\}) = \{a,b\} is open in X, but not closed in X. Therefore f is not strongly continuous.

**Theorem 3.9:** If a map f: X \to Y is strongly α * continuous then it is strongly α *continuous.

**Proof:** Let O be any open set in Y. By [8] \( f^{-1}(O) \) is α *open in X. Therefore f is strongly α *continuous.

**Remark 3.10:** The converse of the above theorem need not be true.

**Example 3.11:** Let X = \{ a, b, c, d \}, \tau = \{ \emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\} \} and \( \sigma = \{ \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\} \} \). Let f: \( (X, \tau) \to (Y, \sigma) \) be defined by f(a) = f(d) = a, f(b) = b, f(c) = c. Clearly, f is not strongly continuous. But f\(^{-1}\)(\{a\}) = \{a,d\} is not open in X. Therefore f is not strongly continuous.

**Theorem 3.12:** If a map f: X \to Y is strongly α * continuous and a map g: Y \to Z is α *continuous then \( g \circ f: X \to Z \) is continuous.

**Proof:** Let O be any open set in Z. Since g is α *continuous, \( g^{-1}(O) \) is α *open in Y. Since f is strongly α *continuous, \( f^{-1}(g^{-1}(O)) \) is open in X. But \( (g \circ f)^{-1}(O) = f^{-1}(g^{-1}(O)) \) is open in X. Therefore, \( g \circ f: X \to Z \) is strongly α *continuous.

**Theorem 3.13:** If a map f: X \to Y is strongly α *continuous and a map g: Y \to Z is α *irresolute, then \( g \circ f: X \to Z \) is strongly α * continuous.

**Proof:** Let O be any α *open set in Z. Since g is α *irresolute, \( g^{-1}(O) \) is α *open in Y. Also, f is strongly α *continuous, \( f^{-1}(g^{-1}(O)) \) is open in X. But \( (g \circ f)^{-1}(O) = f^{-1}(g^{-1}(O)) \) is open in X. Hence, \( g \circ f: X \to Z \) is strongly α *continuous.

**Theorem 3.14:** If a map f: X \to Y is α *continuous and a map g: Y \to Z is strongly α *continuous, then \( g \circ f: X \to Z \) is strongly α *irresolute.

**Proof:** Let O be any α *open set in Z. Since g is strongly α *continuous, \( g^{-1}(O) \) is open in Y. Also, f is strongly α *continuous, \( f^{-1}(g^{-1}(O)) \) is α *open in X. But \( (g \circ f)^{-1}(O) = f^{-1}(g^{-1}(O)) \) is open in X. Hence, \( g \circ f: X \to Z \) is α *irresolute.

**Theorem 3.15:** Let X be any topological spaces and Y be a \( \alpha *T_{1/2} \) space and f: X \to Y be a map. Then the following are equivalent:

1) f is strongly α *continuous
2) f is continuous

**Proof:** (1) \( \Rightarrow \) (2) Let O be any open set in Y. By thm \[O \] is α *open in Y. Then \( f^{-1}(O) \) is open in X. Hence, f is continuous.

(2) \( \Rightarrow \) (1) Let O be any α *open in \( (Y, \sigma) \). Since, \( (Y, \sigma) \) is a \( \alpha *T_{1/2} \) space, O is open in \( (Y, \sigma) \). Since, f is continuous. Then \( f^{-1}(O) \) is open in \( (X, \tau) \). Hence, f is strongly α *continuous.

**Theorem 3.16:** Let f: \( (X, \tau) \to (Y, \sigma) \) be a map. Both \( (X, \tau) \) and \( (Y, \sigma) \) are \( \alpha *T_{1/2} \) space. Then the following are equivalent:

1) f is α *irresolute
2) f is strongly α *continuous
3) f is continuous
4) f is α *continuous
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**Proof:** The proof is obvious.

**Theorem 3.17:** The composition of two strongly $\alpha$ * continuous maps is strongly $\alpha$ * continuous.

**Proof:** Let $O$ be a $\alpha$ * open set in $(Z, \eta)$. Since, $g$ is strongly $\alpha$ * continuous, we get $g^{-1}(O)$ is open in $(Y, \sigma)$. By thm [8] $g^{-1}(O)$ is $\alpha$ * open in $(Y, \sigma)$. As $f$ is also strongly $\alpha$ * continuous, $f^{-1}(g^{-1}(O)) = (g \circ f)^{-1}(O)$ is open in $(X, \tau)$. Hence, $(g \circ f)$ is strongly $\alpha$ * continuous.

**Theorem 3.18:** If $f: (X, \tau) \to (Y, \sigma)$ and $g: (Y, \sigma) \to (Z, \eta)$ be any two maps. Then their composition $g \circ f: (X, \tau) \to (Z, \eta)$ is strongly $\alpha$ * continuous if $g$ is strongly $\alpha$ * continuous and $f$ is continuous.

**Proof:** Let $O$ be a $\alpha$ * open set in $(Z, \eta)$. Since, $g$ is strongly $\alpha$ * continuous, $g^{-1}(O)$ is open in $(Y, \sigma)$. Since $f$ is continuous, $f^{-1}(g^{-1}(O)) = (g \circ f)^{-1}(O)$ is open in $(X, \tau)$. Hence, $(g \circ f)$ is strongly $\alpha$ * continuous.

**IV. Perfectly $\alpha$ * Continuous Function**

**Definition 4.1:** A map $f: (X, \tau) \to (Y, \sigma)$ is said to be perfectly $\alpha$ * continuous if the inverse image of every $\alpha$ * open set in $(Y, \sigma)$ is also perfectly $\alpha$ * open in $(X, \tau)$.

**Theorem 4.2:** If a map $f: (X, \tau) \to (Y, \sigma)$ from a topological space $(X, \tau)$ into a topological space $(Y, \sigma)$ is perfectly $\alpha$ * continuous then it is strongly $\alpha$ * continuous.

**Proof:** Assume that $f$ is perfectly $\alpha$ * continuous. Let $O$ be any $\alpha$ * open set in $(Y, \sigma)$. Since, $f$ is perfectly $\alpha$ * continuous, $f^{-1}(O)$ is open in $(X, \tau)$. Therefore, $f$ is strongly $\alpha$ * continuous.

**Remark 4.3:** The converse of the above theorem need not be true.

**Example 4.4:** Let $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{ab\}, \{bc\}, \{abc\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{c\}, \{ac\}, \{bc\}, \{abc\}, Y\}$. Let $f: (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = a, f(b) = b, f(c) = c$. Clearly, $f$ is perfectly $\alpha$ * continuous. But the inverse image of $\alpha$ * open set in $(Y, \sigma)$ $f^{-1}(\{ac\}) = \{ac\}$ is not open and closed in $X$. Therefore, $f$ is not perfectly $\alpha$ * continuous.

**Theorem 4.5:** If a map $f: (X, \tau) \to (Y, \sigma)$ from a topological space $(X, \tau)$ into a topological space $(Y, \sigma)$ is perfectly $\alpha$ * then it is perfectly continuous.

**Proof:** Let $O$ be an open set in $Y$. By thm [8] $O$ is an $\alpha$ * open set in $(Y, \sigma)$. Since $f$ is perfectly $\alpha$ * continuous, $f^{-1}(O)$ is both open and closed in $(X, \tau)$. Therefore, $f$ is perfectly continuous.

**Remark 4.6:** The converse of the above theorem need not be true.

**Example 4.7:** Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{bc\}, X\}$ and $\sigma = \{\emptyset, \{a\}, Y\}$. Let $f: (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = a, f(b) = b, f(c) = c$. Clearly, $f$ is perfectly continuous. But the inverse image of $\alpha$ * open set in $(Y, \sigma)$ $f^{-1}(\{ac\}) = \{ac\}$ is not open and closed in $X$. Therefore, $f$ is not perfectly $\alpha$ * continuous.

**Theorem 4.8:** A map $f: (X, \tau) \to (Y, \sigma)$ from a topological space $(X, \tau)$ into a topological space $(Y, \sigma)$ is perfectly $\alpha$ * continuous if and only if $f^{-1}(O)$ is both open and closed in $(X, \tau)$ for every $\alpha$ * closed set in $(Y, \sigma)$.

**Proof:** Let $O$ be any $\alpha$ * closed set in $(Y, \sigma)$. Then $O^c$ is $\alpha$ * open in $(Y, \sigma)$. Since, $f$ is perfectly $\alpha$ * continuous, $f^{-1}(O^c)$ is both open and closed in $(X, \tau)$. But $f^{-1}(O^c) = X / f^{-1}(O)$ and so $f^{-1}(O)$ is both open and closed in $(X, \tau)$.

Conversely, assume that the inverse image of every $\alpha$ * closed set in $(Y, \sigma)$ is both open and closed in $(X, \tau)$. Let $O$ be any $\alpha$ * open in $(Y, \sigma)$. Then $O^c$ is $\alpha$ * closed in $(Y, \sigma)$. By assumption $f^{-1}(O^c)$ is both open and closed in $(X, \tau)$. But $f^{-1}(O^c) = X / f^{-1}(O)$ and so $f^{-1}(O)$ is both open and closed in $(X, \tau)$.

**Theorem 4.9:** Let $(X, \tau)$ be a discrete topological space and $(Y, \sigma)$ be any topological space. Let $f: (X, \tau) \to (Y, \sigma)$ be a map, then the following statements are true.

1) $f$ is strongly $\alpha$ * continuous

2) $f$ is perfectly $\alpha$ * continuous

**Proof:** (1) $\Rightarrow$ (2) Let $O$ be any $\alpha$ * open set in $(Y, \sigma)$. By hypothesis, $f^{-1}(O)$ is open in $(X, \tau)$. Since $(X, \tau)$ is a discrete space, $f^{-1}(O)$ is both open and closed in $(X, \tau)$. Hence, $f$ is perfectly $\alpha$ * continuous.

(2) $\Rightarrow$ (1) Let $O$ be any $\alpha$ * open set in $(Y, \sigma)$. Then, $f^{-1}(O)$ is both open and closed in $(X, \tau)$. Hence, $f$ is strongly $\alpha$ * continuous.

**Theorem 4.10:** If $f: (X, \tau) \to (Y, \sigma)$ and $g: (Y, \sigma) \to (Z, \eta)$ are perfectly $\alpha$ * continuous, then their composition $g \circ f: (X, \tau) \to (Z, \eta)$ is also perfectly $\alpha$ * continuous.

**Proof:** Let $O$ be a $\alpha$ * open set in $(Z, \eta)$. Since, $g$ is perfectly $\alpha$ * continuous. We get that $g^{-1}(O)$ is open and closed in $(Y, \sigma)$. By thm [8] $g^{-1}(O)$ is $\alpha$ * open in $(Y, \sigma)$. Since $f$ is perfectly $\alpha$ * continuous, $f^{-1}(g^{-1}(O)) = (g \circ f)^{-1}(O)$ is both open and closed in $(X, \tau)$. Hence, $(g \circ f)$ is perfectly $\alpha$ * continuous.
Theorem 4.11: If \( f: (X, \tau) \to (Y, \sigma) \) and \( g: (Y, \sigma) \to (Z, \eta) \) be any two maps. Then their composition is strongly \( \alpha \ast \) continuous if \( g \) is perfectly \( \alpha \ast \) continuous and \( f \) is continuous.

Proof: Let \( O \) be any \( \alpha \ast \) open set in \((Z, \eta)\). Then, \( g^{-1}(O) \) is open and closed in \((Y, \sigma)\). Since, \( f \) is continuous, \( f^{-1}(g^{-1}(O)) = (g \circ f)^{-1}(O) \) is open in \((X, \tau)\). Hence, \( g \circ f \) is strongly \( \alpha \ast \) continuous.

Theorem 4.12: If a map \( f: (X, \tau) \to (Y, \sigma) \) is perfectly \( \alpha \ast \) continuous and a map \( g: (Y, \sigma) \to (Z, \eta) \) is strongly \( \alpha \ast \) continuous then the composition \( g \circ f: (X, \tau) \to (Z, \eta) \) is perfectly \( \alpha \ast \) continuous.

Proof: Let \( O \) be any \( \alpha \ast \) open set in \((Z, \eta)\). Then, \( g^{-1}(O) \) is open in \((Y, \sigma)\). By Thm [8] \( g^{-1}(O) \) is \( \alpha \ast \) open in \((Y, \sigma)\). By hypothesis, \( f^{-1}(g^{-1}(O)) = (g \circ f)^{-1}(O) \) is both open and closed in \((X, \tau)\). Therefore, \( g \circ f \) is perfectly \( \alpha \ast \) continuous.

V. \( \alpha \ast \) Open maps and \( \alpha \ast \) Closed maps

Definition 5.1: A map \( f: (X, \tau) \to (Y, \sigma) \) is called a \( \alpha \ast \) open if image of each open set in \( X \) is \( \alpha \ast \) open in \( Y \).

Definition 5.2: A map \( f: (X, \tau) \to (Y, \sigma) \) is called a \( \alpha \ast \) closed if image of each closed set in \( X \) is \( \alpha \ast \) closed in \( Y \).

Theorem 5.3: Every closed map is \( \alpha \ast \) closed map.

Proof: The proof follows from the definitions and fact that every closed set is \( \alpha \ast \) closed.

Remark 5.4: The converse of the above theorem need not be true.

Example 5.5: Let \( X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\} \) and \( \sigma = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\} \). Let \( f: (X, \tau) \to (Y, \sigma) \) be defined by \( f(a) = a \), \( f(b) = b \), \( f(c) = c \). Clearly, \( f \) is \( \alpha \ast \) closed but not closed as the image of closed set \{b\} in \( X \) is \{b\} which is not closed set in \( Y \).

Theorem 5.6: Every g-closed map is \( \alpha \ast \) closed.

Proof: Let \( O \) be a closed set in \( X \). Since \( f \) is g-closed map, \( f(O) \) is g-closed in \( Y \). By [8] \( f(O) \) is \( \alpha \ast \) closed in \( Y \). Therefore, \( f \) is \( \alpha \ast \) closed map.

Remark 5.7: The converse of the above theorem need not be true.

Example 5.8: Let \( X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} \) and \( \sigma = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} \). Let \( f: (X, \tau) \to (Y, \sigma) \) be defined by \( f(a) = a \), \( f(b) = b \), \( f(c) = c \). Clearly, \( f \) is \( \alpha \ast \) closed but not \( \alpha \ast \) closed as the image of closed set \{b\} in \( X \) is \{b\} which is not closed set in \( Y \).

Theorem 5.9: Every \( \alpha \ast \) closed map is \( \alpha \ast \) closed.

Proof: The proof follows from the definition and by Thm [8].

Remark 5.10: The converse of the above theorem need not be true.

Example 5.11: Consider \( X = Y = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, X\} \) and \( \sigma = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} \). Let \( f: (X, \tau) \to (Y, \sigma) \) be defined by \( f(a) = a \), \( f(b) = b \), \( f(c) = c \). Clearly, \( f \) is \( \alpha \ast \) closed but not \( \alpha \ast \) closed as the image of closed set \{d\} in \( X \) is \{d\} which is not \( \alpha \ast \) closed set in \( Y \).

Remark 5.12: The composition of \( \alpha \ast \) closed maps need not be \( \alpha \ast \) closed in general as shown in the following example.

Example 5.13: Consider \( X = Y = Z = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, X\} \) and \( \sigma = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} \). Let \( f: (X, \tau) \to (Y, \sigma) \) be defined by \( f(a) = a \), \( f(b) = b \), \( f(c) = c \). Clearly, \( f \) is \( \alpha \ast \) closed. Consider the map \( g: Y \to Z \) defined \( g(a) = a \), \( g(b) = b \), \( g(c) = c \), \( g(d) = d \). Clearly, \( g \) is \( \alpha \ast \) closed. But \( g \circ f: X \to Z \) is not a \( \alpha \ast \) closed, \( g \circ f(|d|) = g(f(|d|)) = g(|d|) = ac \) which is not a \( \alpha \ast \) closed in \( Z \).

Theorem 5.14: A map \( f: (X, \tau) \to (Y, \sigma) \) is \( \alpha \ast \) closed if and only if \( \alpha \ast \mathfrak{cl} (f(A)) \subseteq \mathfrak{cl}(f(A)) \) for each set \( A \) in \( X \).

Proof: Suppose that \( f \) is a \( \alpha \ast \) closed map. Since for each set \( A \) in \( X \), \( \mathfrak{cl}(A) \) is closed set in \( X \), then \( \mathfrak{cl}(f(A)) \) is a \( \alpha \ast \) closed set in \( Y \). Since, \( f(A) \subseteq \mathfrak{cl}(f(A)) \), then \( \alpha \ast \mathfrak{cl}(f(A)) \subseteq \mathfrak{cl}(f(A)) \).

Conversely, suppose \( A \) is a closed set in \( X \). Since \( \alpha \ast \mathfrak{cl}(f(A)) \) is the smallest \( \alpha \ast \) closed set containing \( f(A) \), then \( \mathfrak{cl}(A) \subseteq \alpha \ast \mathfrak{cl}(f(A)) \subseteq \mathfrak{cl}(f(A)) = f(A) \). Thus, \( f(A) = \alpha \ast \mathfrak{cl}(f(A)) \). Hence, \( f(A) \) is a \( \alpha \ast \) closed set in \( Y \). Therefore, \( f \) is a \( \alpha \ast \) closed map.

Theorem 5.15: If \( f: (X, \tau) \to (Y, \sigma) \) is \( \alpha \ast \) closed map and \( g: (Y, \sigma) \to (Z, \eta) \) is \( \alpha \ast \) closed, then the composition \( g \circ f: X \to Z \) is \( \alpha \ast \) closed map.
Proof: Let O be any closed set in X. Since f is closed map, f(O) is closed set in Y. Since, g is α * closed map, g(f(O)) is α * closed in Z which implies that g f(O) = g(f(O)) is α * closed and hence, g f is α * closed.

Remark 5.16: If f: (X, τ) → (Y, σ) is α * closed map and g: (Y, σ) → (Z, η) is closed, then the composition g f: X → Z is not α * closed map as shown in the following example.

Example 5.17: Consider X = Y = Z = {a, b, c, d} , τ = {φ, {a}, {a,b}, X} and σ = {φ, {a}, {a,b}, {a,c}, {a,d}, {b}, {c}, {d}, {a,b,c}, {a,b,d}, {a,c,d}, {b,c,d}}. Let f: (X, τ) → (Y, σ) be defined by f(a) = b, f(b) = c, f(c) = b, f(d) = a. Clearly, f is α * closed. Consider the map g: Y → Z defined g(a) = a, g(b) = b, g(c) = c, g(d) = d. Clearly g is closed. But g f: X → Z is not a α * closed, g f({d}) = g(f({d})) = g(a) = a which is not a α * closed in Z.

Theorem 5.18: Let (X, τ), (Z, η) be topological spaces and (Y, σ) be topological spaces where every α * closed subset is closed. Then the composition g f: (X, τ) → (Z, η) of the α * closed f: (X, τ) → (Y, σ) and g: (Y, σ) → (Z, η) is α * closed.

Proof: Let O be a closed set in X. Since f is α * closed, f(O) is α * closed in Y. By hypothesis, f(O) is closed. Since g is α * closed map, g(f(O)) is α * closed in Z and g f(O) = g f(O) = g f(O). Therefore, g f is α * closed.

Theorem 5.19: If f: (X, τ) → (Y, σ) is g -closed map and g: (Y, σ) → (Z, η) is α * closed and (Y, σ) is a T₁ spaces. Then the composition g f: (X, τ) → (Z, η) is α * closed map.

Proof: Let O be a closed set in X. Since f is g - closed, f(O) is g - closed in (Y, σ) and g is α * closed which implies g f(O) is α * closed in Z and g f(O) = g f(O). Therefore, g f is α * closed.

Theorem 5.20: Let f: (X, τ) → (Y, σ) and g: (Y, σ) → (Z, η) be two mappings such that their composition g f: (X, τ) → (Z, η) be α * closed mapping. Then the following statements are true.

1. If f is continuous and surjective, then g is α * closed.
2. If g is α * - irresolute and injective, then f is α * closed.
3. If f is g - continuous, surjective and f is a T₁ spaces, then g is α * closed.
4. If g is strongly continuous and injective, then f is α * closed.

Proof: 1. Let O be a closed set in (Y, σ). Since f is continuous, f⁻¹(O) is closed in (X, τ). Since g f is α * closed which implies that g f f⁻¹(O) is α * closed in (Z, η). That is g f(O) is α * closed in (Z, η), since f is surjective. Therefore, g is α * closed.

2. Let O be a closed set in (X, τ). Since g f is α * closed, g f(O) is α * closed in (Z, η), since g is α * - irresolute, g⁻¹(g f(O)) is α * closed in (Y, σ). That is f(O) is α * closed in (Y, σ). Since f is injective, Therefore, f is α * closed.

3. Let O be a closed set of (Y, σ). Since, f is g - continuous, f⁻¹(O) is g - closed in (X, τ) and (X, τ) is a T₁ spaces, f⁻¹(O) is closed in (X, τ). Since, g f is α * closed which implies, g f f⁻¹(O) is α * closed in (Z, η). That is g f(O) is α * closed in (Z, η), since f is surjective. Therefore, g is α * closed.

4. Let O be a closed set of (X, τ). Since, g f is α * closed which implies, g f(O) is α * closed in (Z, η). Since, g is strongly α * continuous, g⁻¹(g f(O)) is closed in (Y, σ). That is f(O) is closed in (Y, σ). Since g is injective, f is α * closed.

Theorem 5.21: A map f: (X, τ) → (Y, σ) is α * open if and only if f(int(A)) ⊆ α * int (f(A)) for each set A in X.

Proof: Suppose that f is a α * open map. Since int (A) ⊆ A, then f(int (A)) ⊆ f(A). By hypothesis, f(int (A)) is a α * open and α * int (f(A)) is the largest α * open set contained in f(A). Hence, f(int (A)) ⊆ α * int (f(A)).

Conversely, suppose A is an open set in X. Then f(int(A)) ⊆ α * int (f(A)). Since int (A) = A, then f(A) ⊆ α * int (f(A)). Therefore, f(A) is a α * open set in (Y, σ) and f is α * open map.

Theorem 5.22: Let (X, τ), (Y, σ) and (Z, η) be three topologies spaces f: (X, τ) → (Y, σ) and g: (Y, σ) → (Z, η) be two maps. Then

1. If (g * f) is α * open and f is continuous, then g is α * open.
2. If (g * f) is open and g is α * continuous, then f is α * open map.

Proof: 1. Let A be an open set in Y. Then, f⁻¹(A) is an open set in X. Since (g * f) is α * open map, then (g * f) (f⁻¹(A)) = g f(A) = A is α * open set in Z. Therefore, g is a α * open map.
Let A be an open set in X. Then, g(f(A)) is an open set in Z. Therefore, \( g^{-1}(g(f(A))) = f(A) \) is a \( \alpha^* \) open set in Y. Hence, f is a \( \alpha^* \) open map.

**Theorem 5.23:** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a bijective map. Then the following are equivalent:

1. f is a \( \alpha^* \) open map.
2. f is a \( \alpha^* \) closed map.
3. \( f^{-1} \) is a \( \alpha^* \) continuous map.

**Proof:**

(1) \( \Rightarrow \) (2) Suppose \( O \) is a closed set in X. Then \( X \setminus O \) is an open set in X and by (1) \( f(X \setminus O) \) is a \( \alpha^* \) open in Y. Since, f is bijective, then \( f(X \setminus O) = Y \setminus f(O) \). Hence, \( f(O) \) is a \( \alpha^* \) closed in Y. Therefore, f is a \( \alpha^* \) closed map.

(2) \( \Rightarrow \) (3) Let f is a \( \alpha^* \) closed map and \( O \) be closed set in X. Since, f is bijective then \( f^{-1}(f(O)) = f(O) \) which is a \( \alpha^* \) closed set in Y. Therefore, \( f^{-1} \) is a \( \alpha^* \) continuous map.

(3) \( \Rightarrow \) (1) Let O be an open set in X. Since, \( f^{-1} \) is a \( \alpha^* \) continuous map then \( f^{-1}(f(O)) = f(O) \) is a \( \alpha^* \) open set in Y. Hence, f is a \( \alpha^* \) open map.

**Theorem 5.24:** A map \( f : (X, \tau) \rightarrow (Y, \sigma) \) is \( \alpha^* \) open if and only if for any subset \( O \) of \( (Y, \sigma) \) and any closed set of \( (X, \tau) \) containing \( f^{-1}(O) \), there exists a \( \alpha^* \) closed set \( A \) of \( (Y, \sigma) \) containing \( O \) such that \( f^{-1}(A) \subseteq F \).

**Proof:** Suppose \( f \) is \( \alpha^* \) open. Let \( O \subseteq Y \) and \( F \) be a closed set of \( (X, \tau) \) such that \( f^{-1}(O) \subseteq F \). Now \( X \setminus F \) is an open set in \( (X, \tau) \). Since \( f \) is \( \alpha^* \) open, \( f(X \setminus F) \) is \( \alpha^* \) open in \( (Y, \sigma) \). Then, \( A = Y \setminus f(X \setminus F) \) is a \( \alpha^* \) closed set in \( (Y, \sigma) \). Note that \( f^{-1}(O) \subseteq F \) implies \( O \subseteq A \) and \( f^{-1}(A) = X \setminus f^{-1}(X \setminus F) \subseteq X \setminus (X \setminus F) = F \). That is, \( f^{-1}(A) \subseteq F \).

Conversely, let \( B \) be an open set in \( (X, \tau) \). Then, \( f^{-1}((f(B))^c) \subseteq B^c \) and \( B^c \) is a closed set in \( (X, \tau) \). By hypothesis, there exists a \( \alpha^* \) closed set \( A \) of \( (Y, \sigma) \) such that \( (f(B))^c \subseteq A \) and \( f^{-1}(A) \subseteq B^c \) and so \( B \subseteq (f^{-1}(A))^c \). Hence, \( A^c \subseteq f(B) \subseteq f((f^{-1}(A))^c) \) which implies \( f(B) = A^c \). Since, \( A^c \) is a \( \alpha^* \) open, \( f(B) \) is \( \alpha^* \) open in \( (Y, \sigma) \) and therefore \( f \) is \( \alpha^* \) open map.

References

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