\(\alpha - \) Generalized & \(\alpha^* - \) Separation Axioms for Topological Spaces

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Abstract: The present paper introduces a new class of separation axioms called \(\alpha\)-generalized separation axioms using \(\alpha\)-generalized open sets and also includes the study of the connections between these separation axioms and the existing \(\alpha\)-separation axioms. Also, here, the concept of \(\alpha^*\) - closed set has been coined and then \(\alpha^*\) - separation axioms have been framed w.r.t \(\alpha^*\) - open sets.

Key Words: \(\alpha\)-open sets, \(\alpha^*\) -closed sets, \(\alpha\)-continuous & \(\alpha^*\) - continuous / irresolute functions, \(\alpha^*\) -\(T_k\)\((k=0,1,2)\) and ag-\(T_k\)\((k=0,1,2)\).

I. Introduction

In the mathematical paper [3] O.Njastad introduced and defined an \(\alpha\)-open/closed set. After the works of O.Njastad on \(\alpha\)-open sets, various mathematicians turned their attention to the generalizations of various concepts in topology by considering semi-open, \(\alpha\)-open sets. The concept of \(g\)-closed [1], \(s\)-open [2] and \(\alpha\)-open [3] sets has a significant role in the generalization of continuity in topological spaces. The modified form of these sets and generalized continuity were further developed by many mathematicians [4,5].

In 1970, Levine generalized the concept of closed sets to generalized closed sets[1]. After that there is a vast progress occurred in the field of generalized open sets (compliment of respective closed sets) which became the base for separation axioms in the respective context. In this paper, we introduce the generalized forms of \(\alpha\) - separation axioms using the concepts of \(\alpha\)-generalized open sets called \(\alpha\)-generalized – \(T_k\) (briefly denoted by ag-\(T_k\))spaces. Also, we define the concepts of \(\alpha^*\) - open sets in a topological space in order to frame the another class of separation axioms called \(\alpha^*\) - separation axioms. Among other things, the concern basic properties and relative preservation properties of these spaces are projected under \(\alpha^*\)-irresolute and \(\alpha^*\) - continuous mappings.

II. Preliminaries

Throughout this paper clA and intA respectively closure and the interior of the set A where A is a subset of a topological space \((X, \tau)\) on which no separation axioms are assumed unless explicitly stated. The following definitions and results are listed because of their use in the sequel.

Definition 2.1: Let A be a subset of a space X then A is said to be:

(i) A pre-open if \(A \subseteq \text{int}clA\),

(ii) a semi-open if \(A \subseteq \text{clint}A\),

(iii) \(\alpha\) -open set if \(A \subseteq \text{intcl}A\).

Definition 2.2: (i) The \(\alpha\) -closure of a subset A of X is the intersection of all \(\alpha\) -closed sets that contains A and is denoted by \(\alpha clA\).

(ii) The \(\alpha\) -interior of a subset A of X is the union of all \(\alpha\) -open subsets of X that contained in A and is denoted by \(\alpha intA\).

Definition 2.3: If A be a subset of a space X then A is said to be

(i) an \(\alpha\)-generalized closed (i.e. \(ag\)-closed) set[6] if \(\alpha clA \subseteq U\) whenever \(A \subseteq U\) and \(U\) is an open set,

(ii) a \(\alpha\)-closed \(\alpha\) -closed (i.e. \(ga\)-closed) set[6] if \(\alpha clA \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open set in X,

(iii) an \(\alpha^*\) -closed set if \(\alpha clA \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(ag\)-open set.

Also (iv) \(\alpha\)-generalized closure of a subset A of a space X is the intersection of all \(ag\)-closed sets containing A and is denoted by \(agclA\).

(v) A function \(f: (X, \tau) \rightarrow (Y, \sigma)\) is said to be:

\(\alpha\) - continuous if \(f^{-1}(V)\) is \(\alpha\) - open set in \((X, \tau)\) for every open set \(V\) of \((Y, \sigma)\),

\(g\)-continuous if \(f^{-1}(V)\) is \(g\) - closed in \((X, \tau)\) for every closed set \(V\) of \((Y, \sigma)\),

\(\alpha\) - irresolute if \(f^{-1}(V)\) is \(\alpha\) - open in \((X, \tau)\) for every \(\alpha\) - open set \(V\) of \((Y, \sigma)\).
Proof:
Let x and y be any two distinct points of α–T₀ space X. We show that acl{x} ≠ acl{y}: By hypothesis, suppose that U ∈ α O(x) such that x ∈ U and y ∉ U. Hence y ∈ X – U and X – U is α -closed set. Therefore, acl{y} ⊂ X – U. Hence y ∈ acl{y} as x ∈ X – U.
Hence acl{x} ≠ acl{y}.

Conversely, suppose for any x, y ∈ X with x ≠ y, acl{x} ≠ acl{y}. Without any loss of generality, let z ∈ X such that z ∈ acl{x} but z ∉ acl{y}. Now, we claim that x ∉ acl{y}. For if x ∈ acl{y} then {x} ⊂ acl{y} which implies that acl{x} ⊂ acl{y}. This contradicts the fact that z ∉ acl{y}. Consequently x belongs to the α-open set acl{y} ⊂ which to which y does not belong.
Hence, the space is an α – T₀ space.

Theorem 2.6: A space (X, τ) is α – T₁ iff the singletons are α-closed sets.

Proof: Let x, y ∈ X with x ≠ y. Since X is α – T₁, there exist disjoint α-open sets Y and V in X such that x ∈ Y and y ∈ V, Y ∩ V = φ. Here, U, V ∈ T¹, so, obviously (X, T₀) ceases to be a T₂-space i.e. a Hausdorff space.

Conversely, whenever (X, T²) is a T₂-space, there exist a pair of members of T², say, P & Q for a pair of distinct points p, q of X such that p ∈ P & q ∈ Q & P ∩ Q = φ. But α o(X, T) = T². Combing all these facts (X, T) is α–T₂ space.

Theorem 2.7: Every open subspace of a α–T₂ space is α–T₂.

Proof: Let U be an open subspace of a α–T₂ space (X, τ). Let x and y be any two distinct points of U. Since X is a α–T₂ and U ⊂ X, there exist two disjoint α–open sets G and H in X such that x ∈ G and y ∈ H. Let A = U ∩ G and B = U ∩ H. Then A and B are α-open sets in U containing x and y. Also, A ∩ B = φ.

Hence (U, τ|U) is α–T₂.

* The contents of the preliminaries have been prepared with the help of the paper produced by M. Caldas et al [7].

III. Invariant property of α–Tₖ spaces (k = 0, 1, 2).

We, now, enunciate the invariant property of the α–Tₖ spaces in the following manner:

Theorem 3.1: If f : X → Y be an injective α - irresolute mapping and Y is an α - T₀, then X is α - T₀.

Proof: Let x, y ∈ X with x ≠ y. Since f is injective and Y is α - T₀, there exists a α - open set Vₓ in Y such that f(x) ∈ Vₓ and f(y) ∉ Vₓ or there exists an α-open set Vᵧ in Y such that f(y) ∈ Vᵧ and f(x) ∉ Vᵧ with f(x) ≠ f(y). By α - irresoluteness of f, f⁻¹(Vₓ) is a α-open set in X such that x ∈ f⁻¹(Vₓ) and y ∉ f⁻¹(Vₓ) or f⁻¹(Vᵧ) is a α-open set in X such that y ∈ f⁻¹(Vᵧ) and x ∉ f⁻¹(Vᵧ). This shows that X is α – T₀.

Theorem 3.2: If f : X → Y be an injective α - irresolute mapping and Y is an α – T₁ then X is α – T₁.

Proof: The argument exists in the similar way as mentioned in theorem 3.1 with suitable changes.

Theorem 3.3: If f : X → Y be an injective α - irresolute mapping and Y is an
\( \alpha - T_2 \) then \( X \) is \( \alpha - T_2 \).

**Proof:** Similar is the way as in theorem 3.1 for the establishment of the statement of the theorem under proper changes according to the context.

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**Theorem 4.5:**

The following theorems are.

**Theorem 4.4**

Clearly, every \( \alpha \) distinct points \( \alpha \) is called \( \alpha \) - closed 1-1 mapping.

**Definition 4.1:**

This means that \( \alpha \) - closed 1-1 implies that \( \alpha \) - closed 1-1 mapping.

**Theorem 3.5:**

Sufficiency:

For a pair of distinct points \( x,y \) \( X \), we have as \( \{x\},\{y\} \) \( T \)-closed sets & \( \{f(x),\{f(y)\}\} \) as \( \sigma \)-closed spaces. Therefore, \( \alpha \) - closed 1-1 mapping.

**Theorem 3.7:**

\( \alpha \)-closed 1-1 mapping \( f : X \to Y \) from \( \alpha - T_0 \) space \( X \) into \( \alpha - T_0 \) space \( Y \) exists iff \( f \) is 1-1.

**Proof:**

The necessity follows from the fact mentioned in theorem 3.1. For sufficiency, let \( f : X \to Y \) from \( \alpha - T_0 \) space \( X \) into \( \alpha - T_0 \) space \( Y \) be an one-one mapping. Now for every pair of distinct points \( x,y \in X \), \( \alpha \) - closed 1-1 mapping \( f(\alpha \) - closed 1-1 implies that \( \alpha \) - closed 1-1 mapping.

**Theorem 3.6:**

\( \alpha \) - closed 1-1 mapping \( f : X \to Y \) from \( \alpha - T_0 \) space \( X \) into \( \alpha - T_0 \) space \( Y \) exists iff \( f \) is 1-1.

**Proof:**

Necessity:

For a pair of distinct points \( x,y \in X \), we have as \( \{x\},\{y\} \) \( T \)-closed sets & \( \{f(x),f(y)\}\) as \( \sigma \)-closed spaces. Therefore, \( \alpha \) - closed 1-1 mapping.

This means that \( \alpha \) - closed 1-1 mapping.

**IV.**

\( \alpha \) - Generalized Separation Axioms

Separation axioms using \( \alpha \)-open sets and being weaker than \( \alpha \)-separation axiom are, here, framed due to the motivation of the existence & wide application of \( \alpha \)-open sets.

**Definition 4.1:**

A space \( X \) is called \( \alpha \)-generalized \( - \) \( T_0 \) (briefly written as \( \alpha \) - \( T_0 \)) iff to each pair of distinct points \( x,y \) \( X \), there exists a \( \alpha \)-open set containing one but not the other.

**Definition 4.2:**

A space \( X \) is called \( \alpha \)-generalized \( - \) \( T_1 \) (briefly written as \( \alpha \) - \( T_1 \)) iff to each pair of distinct points \( x,y \) \( X \), there exists a pair of \( \alpha \)-open sets, one containing \( x \) but not \( y \), and the other containing \( y \) but not \( x \).

**Definition 4.3:**

A space \( X \) is called \( \alpha \)-generalized \( - \) \( T_2 \) (briefly written as \( \alpha \) - \( T_2 \)) iff to each pair of distinct points \( x,y \) \( X \), there exists a pair of disjoint \( \alpha \)-open sets, one containing \( x \) and the other containing \( y \). Clearly, every \( \alpha - T_2 \) space is \( \alpha \) - \( T_1 \) spaces \((k=0,1,2)\) since every \( \alpha \)-open set is \( \alpha \)-open set.

**The following theorems are related to the characterization & invariance nature for \( \alpha \) - \( T_k \) spaces:**

**Theorem 4.4:**

A space \( X \) is \( \alpha \) - \( T_0 \) iff \( \alpha \) - closed 1-1 mapping.

**Theorem 4.5:**

A space \( (X,\tau) \) is \( \alpha \) - \( T_1 \) iff the singletons are \( \alpha \)-closed sets.

**Theorem 4.6:**

For a space \( (X,\tau) \) the following are equivalent:

(a) \( X \) is \( \alpha \) - \( T_2 \).

(b) The diagonal \( \Delta = \{(x,x) : x \in X\} \) is \( \alpha \) - closed in \( X \times X \).

**Theorem 4.7:**

If \( f : X \to Y \) be an injective \( \alpha \)-irresolute mapping and \( Y \) is an \( \alpha \) - \( T_k \) then \( X \) is \( \alpha \) - \( T_k \) (\( k=0,1,2 \)).

Furthermore, we mention the concept of \( \alpha \) - \( T_{1/2} \) space in the same time of \( T_{1/2} \) space in topology:

**Definition 4.8:**

A space \( (X,\tau) \) is called an \( \alpha \) - \( T_{1/2} \) space if every \( \alpha \) - closed set is \( \alpha \)-closed.

**Definition 4.9:**

In a topological space \( (X,\tau) \), the following notions are well defined as:

(a) \( \alpha D(X,\tau) = \{A : A \subset X \text{ and } \alpha \text{ is } \alpha \text{-closed} \text{ in } (X,\tau)\} \).

(b) \( \alpha cl^*(E) = \bigcap \{A : E \subset A \in (\alpha \alpha D(X,\tau))\} \).

(c) \( \alpha O(X,\tau) = \{B : \alpha cl^*(B^c) = B^c\} \).
**Theorem 4.10** A topological space \((X, \tau)\) is a \(\alpha^-\) -T\(_{1/2}\) space if and only if \(\alpha O(X, \tau) = \alpha O(X, \tau)^*\) holds.

**Proof. Necessity:**
Let \((X, \tau)\) be a topological space which is also \(\alpha^-\) -T\(_{1/2}\) space. This means that the \(\alpha\)-closed sets and the \(\alpha\)-generalized closed sets coincide by the assumption, \(\alpha cl(E) = \alpha cl^*(E)\) holds for every \(\alpha\)-closed subset \(E\) of \((X, \tau)\). Hence, we have \(\alpha O(X, \tau) = \alpha O(X, \tau)^*\).

**Sufficiency:** Let \(A\) be a \(\alpha\) -closed set of \((X, \tau)\). Then, we have
\[A = \alpha cl^*(A) \text{ & by the accepted criteria } \alpha O(X, \tau) = \alpha O(X, \tau)^*,\]
we claim that \(\alpha cl(A)\) is \(\alpha\)-closed. Therefore \((X, \tau)\) fulfills the criteria for being \(\alpha^-\) -T\(_{1/2}\).

**Theorem 4.11.** A topological space \((X, \tau)\) is a \(\alpha^-\) -T\(_{1/2}\) space if and only if, for each \(x \in X\), \(\{x\}\) is \(\alpha\)-open or \(\alpha\)-closed.

**Proof.** Let topological space \((X, \tau)\) be a \(\alpha^-\) -T\(_{1/2}\) space.

**Necessity:** Let topological space \((X, \tau)\) be a \(\alpha^-\) -T\(_{1/2}\) space. Let us Suppose that for some \(x \in X\); \(\{x\}\) is not \(\alpha\)-closed. Since \(X\) is the only \(\alpha\)-open set containing \(\{x\}\), the set \(\{x\}\) is \(\text{ag}^-\) closed [Definition 2.3] and so it is \(\alpha\)-closed in the \(\alpha^-\) -T\(_{1/2}\) space \((X, \tau)\). Therefore \(\{x\}\) is \(\alpha\)-open.

**Sufficiency:** Since \(\alpha O(X, \tau) \subseteq \alpha O(X, \tau)^*\) holds, it is enough to prove that \(\alpha O(X, \tau)^* \subseteq \alpha O(X, \tau)\). Let \(E \subseteq \alpha O(X, \tau)\). Suppose that \(E \not\subseteq \alpha O(X, \tau)\). Then, \(\alpha cl^*(E^c) = E^c\) and \(\alpha cl(E^c) \neq E^c\) hold. There exists a point \(x\) such that \(x \in E\) and \(x \not\in \alpha Cl(E^c)\).

**Case 1:** \(x\) is \(\alpha\)-open: Since \(\{x\}^c\) is a \(\alpha\)-closed set with \(E^c \subseteq \{x\}^c\), we have \(\alpha Cl(E^c) \subseteq \{x\}^c\). This contradicts the fact that \(x \not\in \alpha Cl(E^c)\).

**Case 2:** \(x\) is \(\alpha\)-closed: Since \(\{x\}^c\) is a \(\alpha\)-open set containing \(\text{ag}^-\) closed set \(A\supseteq E^c\), we have \(\{x\}^c \supseteq \alpha Cl(A) \supseteq \alpha Cl(E^c)\). Therefore \(x \not\in \alpha Cl(E^c)\). This is a contradiction. Therefore \(E \subseteq \alpha O(X, \tau)\).

Hence in both cases, we have \(E \subseteq \alpha O(X, \tau)\). Therefore \(\alpha O(X, \tau)^* \subseteq \alpha O(X, \tau)\).

\[\therefore \alpha O(X, \tau) = \alpha O(X, \tau)^*\] using theorem (3.1), it follows that \((X, \tau)\) is a \(\alpha^-\) -T\(_{1/2}\).

**Theorem 4.12.** A topological space \((X, \tau)\) is a \(\alpha^-\) -T\(_{1/2}\) space if and only if only if, every subset of \(X\) is the intersection of all \(\alpha\)-open sets and all \(\alpha\)-closed sets containing it.

**Proof. Necessity:** Let topological space \((X, \tau)\) be a \(\alpha^-\) -T\(_{1/2}\) space with
\[B \subset X\] arbitrary. Then \(B = \bigcap\{\{x\}^c : x \not\in B\}\) is an intersection of all \(\alpha\)-open sets and \(\alpha\)-closed sets by Theorem 4.11. So the necessity follows.

**Sufficiency:** For each \(x \in X\), \(\{x\}^c\) is the intersection of all \(\alpha\)-open sets and all \(\alpha\)-closed sets containing it. Thus \(\{x\}^c\) is either \(\alpha\)-open or \(\alpha\)-closed and hence \(X\) is \(\alpha^-\) -T\(_{1/2}\) space.
**Definition 5.4(a):** The set of all $\alpha^*$-limit points of $A$, denoted by $\alpha^*(A)$, is called $\alpha^*$-derived set of $A$.

**Definition 5.5:** A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be:

(a) $\alpha^*$-continuous if $f^{-1}(V)$ is $\alpha^*$-open set in $(X, \tau)$ for every open set $V$ of $(Y, \sigma)$.

(b) $\alpha^*$- irresolute if $f^{-1}(V)$ is $\alpha^*$-open in $(X, \tau)$ for every $\alpha^*$-open set $V$ of $(Y, \sigma)$.

**Definition 6:** A space $X$ is called $\alpha^*$-T$_0$ iff to each pair of distinct points $x, y$ of $X$, there exists an $\alpha^*$-open set containing one but not the other.

**Definition 7:** A space $X$ is called $\alpha^*$-T$_1$ space iff to each pair of distinct points $x, y$ of $X$, there exists a pair of $\alpha^*$-open sets, one containing $x$ but not $y$, and the other containing $y$ but not $x$.

**Definition 8:** A space $X$ is called $\alpha^*$-T$_2$ space iff to each pair of distinct points $x, y$ of $X$ there exists a pair of disjoint $\alpha^*$-open sets, one containing $x$ and the other containing $y$.

Clearly, every $\alpha$-T$_k$ space is $\alpha^*$-T$_k$ and $\alpha^*$-T$_k$ space is $\alpha$-T$_k$ space. Since every $\alpha$-open set is $\alpha^*$-open and every $\alpha^*$-open set is $\alpha$-open set where $k = 0, 1, 2$.

**The following theorems are related to the characterization & invariance nature for $\alpha^*$- T$_k$ spaces:**

**Theorem 5.9:** A space $X$ is $\alpha^*$-T$_0$ iff $\alpha^* cl\{x\} \neq \alpha cl\{y\}$ for every pair of distinct points $x, y$ of $X$.

**Theorem 5.10:** A space $(X, \tau)$ is $\alpha^*$-T$_1$ iff the singletons are $\alpha^*$-closed sets.

**Theorem 5.11:** For a space $(X, \tau)$ the following are equivalent:

(a) $X$ is $\alpha^*$-T$_2$.

(b) The diagonal $\Delta = \{(x,x) : x \in X\}$ is $\alpha^*$-closed in $X \times X$.

**Theorem 5.12:** If $f : X \rightarrow Y$ be an injective $\alpha^*$-irresolute mapping and $Y$ is an $\alpha^*$-T$_k$ then $X$ is $\alpha^*$-T$_k$ ($k = 0, 1, 2$).

**Furthermore, we mention the concept of $\alpha^*$-T$_{1/2}$ space in the same tune of $T_{1/2}$ space in topology:**

**Definition 5.12:** A space $(X, \tau)$ is called an $\alpha^*$-T$_{1/2}$ space if every $\alpha^*$-closed set is $\alpha^*$-closed.

The following theorem appears as an evidence for the validity of the above definition of an $\alpha^*$-T$_{1/2}$ space.

**Theorem 5.13:** A topological space $(X, \tau)$ is an $\alpha^*$-T$_{1/2}$ space if and only if, every subset of $X$ is the intersection of all $\alpha^*$-open sets and all $\alpha^*$-closed sets containing it.

**Proof:** The proof is based upon the definitions of the related terms used in the theorem.

**VI. Conclusion**

Separation axioms in terms of $\alpha$-$g$-open & $\alpha^*$-$g$-open sets have been formulated and their structural properties have also been discussed and emphasized which opens the future scope of respective normal and regular topological spaces.

**References**


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