Generalized Contraction Principle in Complex valued Metric spaces

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Abstract: In this paper, we introduce the notion of Generalized contractive type mappings in complex valued metric space and establish fixed point theorem for these mappings.

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I. Introduction

The existence and uniqueness of fixed point theorems of operators or mappings has been a subject of great interest since the work of Banach in 1922 [2]. The Banach contraction mapping principle is widely recognized as the source of metric fixed-point theory. A mapping T : X → X, where (X, d) is metric space, is said to be contraction mapping if for all x, y ∈ X, d(Tx, Ty) ≤ λd(x, y), 0 < λ < 1. (1) According to the Banach contraction mapping principle, any mapping T satisfying (1) in Complete metric space will have a unique fixed-point. This principle includes different directions in different spaces adopted by mathematicians; for example, metric space, G-metric spaces, partial metric spaces, cone metric spaces have already been obtained. A new space called the complex valued metric space which is more general than well-known metric space has been introduced by Azam et al. Azam proved some fixed-point theorems for mappings satisfying a rational inequality. In 2012, Rouzkard and Imdad [3] extended and improved the common fixed-point theorems which are more general than the result of Azam et al. [1]. The idea of complex valued metric spaces can be exploited to define complex valued normed spaces and complex valued Hilbert spaces; additionally, it offers numerous research activities in mathematical analysis.

II. Basic Facts and Definitions

We recall some notations and definitions which will be utilized in our discussion.

Let C be a set of complex numbers and z₁, z₂ ε C. Define a partial order ≼ on C as follows:

z₁ ≼ z₂ if and only if Re(z₁) ≤ Re(z₂) and Im(z₁) ≤ Im(z₂).

It follows that z₁ ≼ z₂ if one of the following conditions is satisfied:

(i) Re(z₁) = Re(z₂) and Im(z₁) ≤ Im(z₂).
(ii) Re(z₁) < Re(z₂) and Im(z₁) = Im(z₂).
(iii) Re(z₁) < Re(z₂) and Im(z₁) < Im(z₂).
(iv) Re(z₁) > Re(z₂) and Im(z₁) = Im(z₂).

In (i), (ii) and (iii), we have |z₁| < |z₂|. In (iv), we have |z₁| > |z₂|. Therefore, we write z₁ ≼ z₂ and one of (i), (ii) and (iii) is satisfied. In this case |z₁| < |z₂|. We will write z₁ ≼ z₂ if and only if (iii) is satisfied.

Take into account some fundamental properties of the partial order ≼ on C as follows.

(i) If 0 ≤ z₁ ≼ z₂, then |z₁| < |z₂|.
(ii) If z₁ ≼ z₂ and z₂ ≼ z₃, then z₁ ≼ z₃.
(iii) If z₁ ≼ z₂ and λ > 0 is a real number, then λz₁ ≼ λz₂.

Definition 1 [3]: The "max" function for the partial order relation "≽" is defined by the following.

(i) max{z₁, z₂} = z₁ if and only if z₁ ≼ z₂.
(ii) If z₁ ≼ max{z₁, z₂}, then z₁ ≤ z₂ or z₁ ≼ z₂.
(iii) max{z₁, z₂} = z₂ if and only if z₁ ≼ z₂ and |z₁| < |z₂|.

Using Definition 1 one can have the following lemma.

Lemma 2 [3]: Let z₁, z₂, z₃,... ∈ C and the partial order relation ≼ is defined on C. Then the following conditions are easy follow.

(i) If z₁ ≼ max{z₃, z₄}, then z₁ ≼ z₃ if z₁ ≼ z₄.
(ii) If z₁ ≼ max{z₃, z₄} and z₁ ≼ z₃, then z₁ ≤ z₄ if z₁ ≼ z₄.
(iii) If z₁ ≼ max{z₃, z₄, z₅}, then z₁ ≤ z₃ if max{z₃, z₄, z₅} ≼ z₅.
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Definition 3.1. Let \( X \) be non empty set. If a mapping \( d: X \times X \rightarrow C \) satisfies

(i) \( 0 \leq d(x, y) \) for all \( x, y \in X \) and \( d(x, y)=0 \) if and only if \( x=y \),

(ii) \( d(x, y) \leq d(y, x) \) for all \( x, y \in X \),

(iii) \( d(x, y) \leq d(x, z) + d(z, y) \) for all \( x, y, z \in X \).

Then \( d \) is called a complex valued metric on \( X \) and the pair \((X, d)\) is called complex valued metric space.

Let \( \{x_n\} \) be a sequence in complex valued metric space \( X \) and \( x \in X \). If for every \( \varepsilon \in C \) with \( 0 < \varepsilon \) there \( N \in \mathbb{N} \) such that , for all \( n > N \), \( d(x_n, x) < \varepsilon \), then \( \{x_n\} \) is called a Cauchy sequence in \( X \). If every Cauchy sequence is convergent in \( X \), then \( X \) is called a complete complex valued metric space.

Lemma 2. [1] Let \( (X, d) \) be a complex valued metric space and let \( \{x_n\} \) be a sequence in \( X \). Then

(i) \( \{x_n\} \) converges to \( x \) if and only if \( \left| d(x_n, x) \right| \rightarrow 0 \) as \( n \rightarrow \infty \).

(ii) \( \{x_n\} \) is Cauchy sequence if and only if \( \left| d(x_n, x_m) \right| \rightarrow 0 \) as \( n \rightarrow \infty \).

III. Main Results

In this paper, we prove Generalized contraction principle in complex valued metric space as follows:

Theorem 1.1. Let \( T: X \rightarrow X \) be self mappings of a complex valued metric space \((X, d)\) satisfying

\[ d(Tx, Ty) \leq k d(x, y) \quad \text{where } k \in [0,1) \]  

(2)

where \( M(x) = \max\{d(x, y), d(Tx, y), d(Ty, y), \frac{d(Ty, y) + d(Tx, x)}{2}\} \).

Then \( T \) has a unique fixed point.

Proof. Let \( x \in X \) be arbitrary point and define a sequence \( \{x_n\} \) as \( x_{n+1} = Tx_n \). Then putting \( x = x_n \), \( y = x_{n+1} \) we get

\[ d(x_{n+1}, x_n) = d(Tx_n, x_n) \leq k M(x_n) \]  

(3)

where \( M(x_n) = \max\{d(x_n, x_{n-1}), d(Tx_n, x_n), d(x_{n-1}, x_n), \frac{d(Tx_n, x_n) + d(Tx_{n-1}, x_n)}{2}\} \).

\[ M(x_n) = \max\{d(x_n, x_{n-1}), d(x_{n-1}, x_n), \frac{d(x_n, x_{n-1}) + d(x_{n-1}, x_n)}{2}\} \]

\[ \leq \max\{d(x_{n+1}, x_n), d(x_n, x_{n-1})\} \].

Now from (2) we get

\[ d(x_{n+1}, x_n) = d(Tx_n, x_n) \leq k \max\{d(x_{n+1}, x_n), d(x_n, x_{n-1})\} \].

We shall take two cases.

Suppose \( \left| d(x_{n+1}, x_n) \right| > \left| d(x_n, x_{n-1}) \right| \). Since \( \left| d(x_{n+1}, x_n) \right| > 0 \), we obtain

\[ \left| d(x_{n+1}, x_n) \right| \leq k \left| d(x_{n+1}, x_n) \right| \] a contradiction. Therefore, we get

\[ \max\{d(x_{n+1}, x_n), d(x_n, x_{n-1})\} = \left| d(x_{n+1}, x_n) \right| \].

Then \( d(x_{n+1}, x_n) \leq k \left| d(x_{n+1}, x_n) \right| \).

Again \( d(x_{n+1}, x_n) \leq k \left| d(x_{n+1}, x_n) \right| \).

Continuing in the same manner, we have \( d(x_{n+1}, x_n) \leq k^n \left| d(x_1, x_0) \right| \).

(4)

Then for all \( n, m \in \mathbb{N} \) and repeated use of triangular inequality for \( m \geq 1 \) and from (5), we have

\[ d(x_{n+m}, x_n) \leq d(x_{n+m}, x_{n+m-1}) + \cdots + d(x_{n+1}, x_n) \leq \sum_{p=n}^{m-1} k^p \left| d(x_1, x_0) \right| \]

(5)

Therefore, \( \left| d(x_{n+m}, x_n) \right| \leq \sum_{p=n}^{m-1} k^p \left| d(x_1, x_0) \right| \).

Since \( k \in [0,1) \), if we take limit as \( n \rightarrow \infty \) then \( \left| d(x_{n+m}, x_n) \right| \rightarrow 0 \).

So, \( \{x_n\} \) is complex valued Cauchy sequence. By completeness of \((X, d)\) there exists \( z \in X \) such that \( \{x_n\} \) is complex valued convergent to \( z \).

Next we prove \( Tz = z \). Assume on contrary that \( Tz \neq z \). Then by (1), put \( x = z \), \( y = x_{n+1} \)

\[ d(Tx, Ty) \leq k M(x, x_{n+1}) \]

where \( M(x, x_{n+1}) = \max\{d(z, x_{n+1}), d(Tz, z), d(Tx_{n+1}, x_{n+1}), \frac{d(Tz, x_{n+1}) + d(z, x_{n+1})}{2}\} \).

As \( \{x_n\} \) is convergent to \( z \), therefore, \( \lim_{n \to \infty} \frac{d(z, x_n)}{2} = d(z, x) \) and \( \lim_{n \to \infty} d(x_n, x) = 0 \).

Thus letting \( n \to \infty \), \( d(Tz, z) \leq k d(Tz, z) \) which is contradiction.

So, \( Tz = z \) that is, \( z \) is fixed point of \( T \).

Uniqueness. Let \( u (\neq z) \) be another fixed point of \( T \), then from (2) we have

\[ d(u, z) = d(Tu, Tz) \leq k M(u, z) \]

where \( M(u, z) = \max\{d(u, z), d(Tu, u), d(Tz, z), \frac{d(Tu, z) + d(Tz, u)}{2}\} \).

Conclusion. The generalized contraction principle in complex valued metric spaces is proved.
\[ d(u, z) \leq k \left| d(u, z) \right| \] which is a contradiction. Hence \( u = z \) that is, \( T \) has a unique fixed point.

**References**


