Gaussian -Diophantine quadruples with property D (1)

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Abstract: A set of m Gaussian integers is called a complex Diophantine m-tuple with the property D (z) if the product of its any two distinct elements increased by z is a square of a Gaussian integer. In this paper, we present Gaussian-Diophantine quadruples with property D(1). Few examples of complex Diophantine quadruples with the property D (1) are presented.

Keywords: Diophantine quadruples, Integral solutions, Gaussian integers, Pell equation

I. Introduction

A set of positive integers \{a_1, a_2, a_3, \ldots, a_m\} is said to have the property D (n), if \(a_i a_j + n\) is a perfect square for all \(1 \leq i \leq j \leq m\) and such a set is called a Diophantine m-tuples with property D(n). Many mathematicians considered the problem of the existence of a Diophantine quadruples with property D (n) for any arbitrary integer n[1] and also, for any linear polynomials in n. Further various authors considered the connections of the problem of Diaphanous, Davenport and Fibonacci numbers in [2-21].

In this paper we consider the analogous problem for Gaussian integers. Let z be a Gaussian integer and let \(m \geq 2\) be an integer. A set \(\{a_1, a_2, a_3, \ldots, a_m\} \subset \mathbb{Z}(i) \setminus \{0\}\) is said to have this property D (z) if the product of its any two distinct elements increased by z is a square of a Gaussian integer. If the set \(\{a_1, a_2, a_3, \ldots, a_m\}\) is a complex Diophantine quadruple then the same is true for the set \(\{-a_1, -a_2, -a_3, \ldots, -a_m\}\). Particularly in [22], the authors have analyzed the problem of the existence of the complex Diophantine quadruples. In this paper, we present a Gaussian -Diophantine quadruple with property D (1).

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Method of analysis:

To start with, it is seen that the pair \((a, b)\) is Gaussian Diophantine 2-tuples with property D (1) where \(a\) and \(b\) are Gaussian integers of the form

\[
a = (2p^2 - 2q^2 - p) + i(4pq - q) \quad \text{and} \quad b = (2p^2 - 2q^2 + 7p + 6) + i(4pq + 7q)
\]

Let \(c_s\) be any non zero integer such that

\[
a * c_s + 1 = \alpha_s^2 \quad (1)
\]

\[
b * c_s + 1 = \beta_s^2 \quad (2)
\]

Eliminating \(c_s\) between (1) and (2) we get

\[
b \alpha_s^2 - a \beta_s^2 = b - a \quad (3)
\]

Substitution of the linear transformations

\[
\alpha_s = X_s + aT_s \quad (4)
\]

\[
\beta_s = X_s + bT_s \quad (5)
\]

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in (3) leads to the equation

\[ X_s^2 = abT_s^2 + 1 \]  \hfill (6)

where

\[ ab = (4p^4 + 4q^4 - 24p^2p^2 - 36pq^2 + 12q^3 + 5p^2 - 5q^2 - 6p) + \\
i(16p^3q - 16pq^3 + 36p^2q - 12q^3 + 10pq - 6q) \]

The general solution of (6) is given by

\[
\begin{align*}
X_s &= \frac{1}{2}[(X_0 + \sqrt{abT_0})^{s+1} + (X_0 - \sqrt{abT_0})^{s+1}] \\
T_s &= \frac{1}{2\sqrt{ab}}[(X_0 + \sqrt{abT_0})^{s+1} - (X_0 - \sqrt{abT_0})^{s+1}]
\end{align*}
\]  \hfill (7)

Taking \( s = 0 \) in (7) and using (4) we have

\[ \alpha_0 = (4p^2 - 4q^2 + 2p - 1) + i(8pq - 2q) \]  \hfill (8)

In view of (1) we have

\[ c_0 = (8p^2 - 8q^2 + 12p + 4) + i(16pq + 12q) \]  \hfill (9)

Observe that

\[
\{(2p^2 - 2q^2 - p) + i(4pq - q), (2p^2 - 2q^2 + 7p + 6) + i(4pq + 7q), \\
(8p^2 - 8q^2 + 12p + 4) + i(16pq + 12q)\}
\]

is a Gaussian Diophantine triple with property D(1).

Again taking \( s = 1 \) in (7) and using (4) we obtain

\[ \alpha_1 = (16p^4 + 16q^4 - 88p^2q^2 + 32p^3 - 76pq^2 - 10p - 20pq^2 + 1) + \\
i(64p^3q - 64pq^3 + 96p^2q - 32q^3 - 10q) \]

In view of (1) we have

\[ c_1 = \{128(p^6 - 15p^4q^2 + 15p^2q^4 - q^6) + 576(p^5 - 10p^3q^2 + 5pq^4) + \\
800(p^4 - 6p^2q^2 + q^4) + 240(p^3 - 3pq^2) - 184(p^2 - q^2) - 60p + 20\} + \\
i\{(128(6p^5q - 20p^3q^3 + 6q^5p) + 576(5p^4q - 10p^2q^3 + q^5) + \\
800(4p^3q - 4pq^3) + 240(3p^2q - q^3) - 368pq - 60q\}
\]

Hence

\[
\{(2p^2 - 2q^2 - p) + i(4pq - q), (2p^2 - 2q^2 + 7p + 6) + i(4pq + 7q), \\
(8p^2 - 8q^2 + 12p + 4) + i(16pq + 12q), \\
(128(p^6 - 15p^4q^2 + 15p^2q^4 - q^6) + 576(p^5 - 10p^3q^2 + 5pq^4) + \\
800(p^4 - 6p^2q^2 + q^4) + 240(p^3 - 3pq^2) - 184(p^2 - q^2) - 60p + 20\} + \\
i\{(128(6p^5q - 20p^3q^3 + 6q^5p) + 576(5p^4q - 10p^2q^3 + q^5) + \\
800(4p^3q - 4pq^3) + 240(3p^2q - q^3) - 368pq - 60q\}
\]

is a Gaussian Diophantine quadruple with property D(1).

The repetition of the above process leads to the generation of infinitely many Gaussian Diophantine quadruples with property D(1).
**Table: Examples**

<table>
<thead>
<tr>
<th>((p,q))</th>
<th>Diophantine quadruple with property D(1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0,1))</td>
<td>((-2-i, 4+7i, -4+12i, 876+276i))</td>
</tr>
<tr>
<td>((1,1))</td>
<td>((-1+3i, 13+11i, 16+28i, -6024-3276i))</td>
</tr>
<tr>
<td>((1,2))</td>
<td>((-7+6i, 7+22i, -8+56i, 30864-36792i))</td>
</tr>
</tbody>
</table>

**Note:**

If \(\{z_1, z_2, z_3, z_4\}\) is a quadruple with the property \(D(z)\), then \(\{\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4}\}\) is a quadruple with the property \(D(\overline{z})\).

**II. Conclusion**

In this paper, we have presented a Gaussian Diophantine quadruple with property D(1). One may search for Gaussian Diophantine quadruples consisting of special numbers with suitable properties.

**References**


[22]. Andrej Dujella,Zagreb,Croatia, ”The Problem of Diophantus and Davenport for Gaussian Integers” Glas.Mat.Ser.III 32(1997), 1-10