Total Dominating Sets and Total Domination Polynomials of Square Of Paths

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Abstract: Let $G=\langle V, E \rangle$ be a simple connected graph. A set $S \subseteq V$ is a total dominating set of $G$ if every vertex is adjacent to an element of $S$. Let $D_i(P_n^2, j)$ be the family of all total dominating sets of the graph $P_{n}^2$, $n \geq 2$ with cardinality $i$, and let $d_i(P_n^2, j) = |D_i(P_n^2, j)|$. In this paper we computed $d_i(P_n^2, j)$, and obtain the polynomial $D_i(P_n^2, x) = \sum_{i=1}^{n} d_i(P_n^2, i) x^i$ which we call total domination polynomial of $P_n^2$, $n \geq 2$ and obtain some properties of this polynomial.

Keywords: total domination set, total domination polynomial, square of path

I. Introduction

Let $G=\langle V, E \rangle$ be a simple connected graph. A set $S \subseteq V$ is a dominating set of $G$, if every vertex in $V-S$ is adjacent to at least one vertex in $S$. A set $S \subseteq V$ is total dominating set if every vertex of the graph is adjacent to an element of $S$. The total domination number of a graph $G$ is the minimum cardinality of a total dominating set in $G$, and it is denoted by $\gamma_t(G)$. Obviously $\gamma_t(G) < |V|$. A path is a connected graph in which two vertices have degree 1 and the remaining vertices have degree 2. Let $P_n$ be the path with $n$ vertices. The square of a simple connected graph $G$ is a graph with same set of vertices of $G$ and an edge between two vertices if and only if there is a path of length at most 2 between them. It is denoted by $G^2$. We use the notation $\lfloor x \rfloor$ for the largest integer less than or equal to $x$ and $\lceil x \rceil$ for the smallest integer greater than or equal to $x$. Also we denote the set $\{1, 2, \ldots, n\}$ by $[n]$, throughout this paper.

Let $P_n^2$, $n \geq 2$ be the square of the path $P_n$, $n \geq 2$ and let $D_i(P_n^2, i)$ be the family of total dominating sets of the graph $P_n^2$, $n \geq 2$ with cardinality $i$, and let $d_i(P_n^2, i) = |D_i(P_n^2, i)|$.

Table 1 in page number 12 gives the number of total dominating sets of $P_n^2$, with cardinality $i$. The total domination polynomial $D_i(P_n^2, x)$ of $P_n^2$, $n \geq 2$ is defined as $D_i(P_n^2, x) = \sum_{i=1}^{n} d_i(P_n^2, i) x^i$, where $\gamma_t(P_n^2)$ is the total domination number of $P_n^2$, $n \geq 2$.

II. Total Dominating Sets of Square Of Paths

Let $D_i(P_n^2, i)$ be the family of total dominating sets of $P_n^2$, $n \geq 2$ with cardinality $i$. We will investigate total dominating sets of $P_n^2$, $n \geq 2$.

Lemma 2.1

$$\gamma_t(P_n^2) = \begin{cases} \lfloor \frac{n}{5} \rfloor + 2 & \text{if } n \equiv 5(\text{mod} 5) \\ \lceil \frac{n}{5} \rceil + 1 & \text{if } n \not\equiv 5(\text{mod} 5) \end{cases}$$

Lemma 2.2

Let $P_n^2$, $n \geq 2$ be the square of path $P_n$, with $|V(P_n^2)| = n$. Then $d_i(P_n^2, i) = 0$ if $i < \left\lfloor \frac{n}{5} \right\rfloor + 1$ or $i > n$ and $d_i(P_n^2, i) > 0$ if $\left\lfloor \frac{n}{5} \right\rfloor + 1 \leq i \leq n$.

Proof:

If $n \equiv 5(\text{mod} 5)$, then the total domination number of the square of path $P_n^2$ is $\gamma_t(P_n^2) = \left\lfloor \frac{n}{5} \right\rfloor + 2$. Therefore $d_i(P_n^2, i) = 0$ if $i < \left\lfloor \frac{n}{5} \right\rfloor + 2$ or $i > n$ and $d_i(P_n^2, i) > 0$ if $\left\lfloor \frac{n}{5} \right\rfloor + 2 \leq i \leq n$.

On the other hand, if $n \not\equiv 5(\text{mod} 5)$, then the total domination number of $P_n^2$ is $\gamma_t(P_n^2) = \left\lceil \frac{n}{5} \right\rceil + 1$. Therefore $d_i(P_n^2, i) = 0$ if $i < \left\lceil \frac{n}{5} \right\rceil + 1$ or $i > n$ and $d_i(P_n^2, i) > 0$ if $\left\lceil \frac{n}{5} \right\rceil + 1 \leq i \leq n$.

Hence, on both the cases we have $d_i(P_n^2, i) = 0$ if $i < \left\lfloor \frac{n}{5} \right\rfloor + 1$ or $i > n$ and $d_i(P_n^2, i) > 0$ if $\left\lfloor \frac{n}{5} \right\rfloor + 1 \leq i \leq n$.

Lemma 2.3

Let $P_n^2$, $n \geq 2$ be the square of path with $|V(P_n^2)| = n$. Then we have
Lemma 2.4
Let $P_n^2$, $n \geq 2$ be the square of path with $\left| V (P_n^2) \right| = n$. Then we have

(i) If $D_1(P_n^2, i) = \phi$ or $i < \gamma_i(P_n^2)$ or $i > n$.

(ii) $D_2(P_n^2, x)$ has no constant term and first degree terms.

(iii) $D_3(P_n^2, x)$ is a strictly increasing function on $[0, \infty)$.

Proof of (i)
Since $P_n^2$ has $n$ vertices, there is only one way to choose all these vertices. Therefore $d_i(P_n^2, n) = 1$. Out of these $n$ vertices, every combination of $n-1$ vertices can dominate totally only if $\delta (P_n^2) > 1$.
Therefore $d_i(P_n^2, n) = n$ if $\delta (P_n^2) > 1$.

Therefore $D_1(P_n^2, i) = \phi$ if $i < \gamma_i(P_n^2)$ and $D_1(P_n^2, n+k) = \phi$, $k = 1, 2, 3, \ldots$.

Thus we have $d_i(P_n^2, i) = 0$ or $i < \gamma_i(P_n^2)$ and $d_i(P_n^2, n+i) = 0$, for $i = 1, 2, 3, \ldots$.

Proof of (ii)
A single vertex of $P_n^2$ cannot totally dominate all the vertices of $P_n^2, n \geq 2$. So the set of all vertices of $P_n^2$ is totally dominated by at least two of the vertices of $P_n^2$. Hence the total domination polynomial has no constant term as well as first degree term.

Proof of (iii)
By the definition of total domination, every vertex of $P_n^2$ is adjacent to an element of total domination set.

That is $D_1(P_n^2, x) = \sum_{i=1}^{n} \gamma_i(P_n^2) d_i(P_n^2, i)x^i$

Therefore $D_1(P_n^2, x)$ is a strictly increasing function on $[0, \infty)$.

Lemma 2.5
Let $P_n^2$, $n \geq 2$ be the square of path with $\left| V (P_n^2) \right| = n$. Suppose that $D_1(P_n^2, i) = \phi$, then we have

(i) $D_1(P_{n-2}^2, i-1) = \phi$, $D_2(P_{n-3}^2, i-1) = \phi$ if and only if $n = i$.

(ii) $D_2(P_{n-2}^2, i-1) = \phi$, $D_3(P_{n-3}^2, i-1) = \phi$ and $D_3(P_{n-5}^2, i-1) = \phi$ if and only if $n = 3k + 3$.

(iii) $D_2(P_{n-2}^2, i-1) = \phi$, $D_3(P_{n-3}^2, i-1) = \phi$, $D_3(P_{n-4}^2, i-1) = \phi$, $D_3(P_{n-5}^2, i-1) = \phi$.
(iv) $D_4(P_{n,5}^2,i-1) \neq \varphi$, $D_4(P_{n,2}^2,i-1) \neq \varphi$. $D_4(P_{n,3}^2,i-1) \neq \varphi$. $D_4(P_{n,4}^2,i-1) \neq \varphi$ and $D_4(P_{n,5}^2,i-1) \neq \varphi$ if and only if
$$\left[\frac{n-1}{5}\right] + 2 \leq i \leq n-4$$

**Proof of (i)**

Suppose, $D_4(P_{n,5}^2,i-1) = \varphi$, $D_4(P_{n,3}^2,i-1) = \varphi$, $D_4(P_{n,4}^2,i-1) = \varphi$. $D_4(P_{n,5}^2,i-1) = \varphi$

$\Rightarrow d_i(P_{n,5}^2,i-1) = 0$, $d_i(P_{n,3}^2,i-1) = 0$, $d_i(P_{n,4}^2,i-1) = 0$, $d_i(P_{n,5}^2,i-1) = 0$

$\Rightarrow i-1 < \left[\frac{n-1}{5}\right] + 1$ or $i-1 > n-2$; $i-1 < \left[\frac{n-1}{5}\right] + 1$ or $i-1 > n-3$; $i-1 < \left[\frac{n-1}{5}\right] + 1$ or $i-1 > n-4$ and

$i-1 < \left[\frac{n-5}{5}\right] + 1$ or $i-1 > n-5$.

If $i-1 < \left[\frac{n-5}{5}\right] + 1$ or $i-1 > n-5$, then $d_i(P_{n,5}^2,i-1) = 0$ which is a contradiction, so $d_i(P_{n,5}^2,i-1) \neq 0$. Therefore, $i-1 > n-2 > n-3 > n-4 > n-5$

$\Rightarrow i-1 > n-2 \Rightarrow i-1 > n-1 \Rightarrow i-1 \geq n$

(1)

Also $d_i(P_{n,5}^2,i-1) \neq 0 \Rightarrow \left[\frac{n-1}{5}\right] + 1 \leq i-1 \leq n-1 \Rightarrow i-1 \leq n-1 \Rightarrow i-1 < n \Rightarrow i < n$

(2)

From (1) and (2), we have, $i = n$.

Conversely, if $i = n$, then

$D_4(P_{n,3}^2,i-1) = D_4(P_{n,5}^2,i-1) = \varphi$. $D_4(P_{n,3}^2,i-1) = \varphi$. $D_4(P_{n,4}^2,i-1) = \varphi$. $D_4(P_{n,5}^2,i-1) = \varphi$.

$D_4(P_{n,5}^2,i-1) = \varphi$ and $D_4(P_{n,4}^2,i-1) = \varphi$, since $D_4(P_{n,5}^2,i-1) \neq \varphi$.

**Proof of (ii)**

Suppose, $D_4(P_{n,1}^2,i-1) \neq \varphi$, $D_4(P_{n,2}^2,i-1) \neq \varphi$. $D_4(P_{n,3}^2,i-1) \neq \varphi$ and $D_4(P_{n,4}^2,i-1) \neq \varphi$.

Then, $d_i(P_{n,2}^2,i-1) \neq 0$, $d_i(P_{n,3}^2,i-1) \neq 0$, $d_i(P_{n,4}^2,i-1) \neq 0$ and $d_i(P_{n,5}^2,i-1) \neq 0$.

$\Rightarrow \left[\frac{n-1}{5}\right] + 1 \leq i-1 \leq n-1$; $\left[\frac{n-2}{5}\right] + 1 \leq i-1 \leq n-2$; $\left[\frac{n-3}{5}\right] + 1 \leq i-1 \leq n-3$ and $\left[\frac{n-4}{5}\right] + 1 \leq i-1 \leq n-4$.

Also, $D_4(P_{n,5}^2,i-1) = \varphi$. $d_i(P_{n,5}^2,i-1) = 0 \Rightarrow i-1 < \left[\frac{n-5}{5}\right] + 1$ or $i-1 > n-5$.

If $i-1 < \left[\frac{n-5}{5}\right] + 1$, then $i-1 < \left[\frac{n-5}{5}\right] + 1 \Rightarrow i-1 < \left[\frac{n-5}{5}\right] + 1$.

$\Rightarrow d_i(P_{n,5}^2,i-1) = 0$ which is a contradiction, since $d_i(P_{n,5}^2,i-1) \neq 0 \Rightarrow i-1 < n-3$.

From (1) and (2), we have, $i = n-3$. Therefore, $i-1 < \left[\frac{n-5}{5}\right] + 1$ is not possible, so $i-1 > n-5$

$\Rightarrow i > n-4 \Rightarrow i \geq n-3$

(1)

Since, $d_i(P_{n,3}^2,i-1) \neq 0 \Rightarrow \left[\frac{n-4}{5}\right] + 1 \leq i-1 \leq n-4 \Rightarrow i-1 \leq n-4$.

Conversely, if $i = n-3$, then

$D_4(P_{n,1}^2,i-1) = \varphi$. $D_4(P_{n,2}^2,i-1) = \varphi$. $D_4(P_{n,3}^2,i-1) = \varphi$. $D_4(P_{n,4}^2,i-1) = \varphi$.

$D_4(P_{n,4}^2,i-1) = \varphi$ and $D_4(P_{n,5}^2,i-1) = \varphi$.

**Proof of (iii)**

Suppose, $D_4(P_{n,1}^2,i-1) = \varphi$. $D_4(P_{n,2}^2,i-1) = \varphi$. $D_4(P_{n,3}^2,i-1) = \varphi$

Then, $d_i(P_{n,1}^2,i-1) = 0$, $d_i(P_{n,2}^2,i-1) = 0$ and $d_i(P_{n,3}^2,i-1) = 0$.

$\Rightarrow i-1 < \left[\frac{n-1}{5}\right] + 1$ or $i-1 > n-1$; $i-1 < \left[\frac{n-2}{5}\right] + 1$ or $i-1 > n-2$ and $i-1 < \left[\frac{n-3}{5}\right] + 1$ or $i-1 > n-3$.

If $i-1 > n-3$ or $n-2 > n-5$, then $D_4(P_{n,3}^2,i-1) = \varphi$. $D_4(P_{n,4}^2,i-1) = \varphi$.

Therefore, $i-1 > n-3$ is not possible. So, $i-1 < \left[\frac{n-3}{5}\right] + 1$ or $i-1 < \left[\frac{n-1}{5}\right] + 1$ or $i-1 < \left[\frac{n-1}{5}\right] + 2$ or $i < \left[\frac{n-1}{5}\right] + 1$.

Also, $\left[\frac{n-1}{5}\right] + 2 \leq \left[\frac{n-2}{5}\right] + 2 \Rightarrow i < \left[\frac{n-2}{5}\right] + 2$.

Therefore, $i-1 > n-2$ and $i = k-2$ for some positive integer $k$.

Then, $D_4(P_{n-1}^2,i-1) = \varphi$. $D_4(P_{n-2}^2,i-1) = \varphi$. $D_4(P_{n-3}^2,i-1) = \varphi$. $D_4(P_{n-4}^2,i-1) = \varphi$. $D_4(P_{n-5}^2,i-1) = \varphi$.

**Proof of (v)**

Suppose $D_4(P_{n,1}^2,i-1) = \varphi$. $D_4(P_{n,2}^2,i-1) = \varphi$. $D_4(P_{n,3}^2,i-1) = \varphi$. $D_4(P_{n,4}^2,i-1) = \varphi$ and $D_4(P_{n,5}^2,i-1) = \varphi$.

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When $n = 2$ or $n = 3$. When $n = 2$, n = 3 or n = 4. When $n = 2$ or $n = 3$, then adjoin $n - 1$ with $X_2$. Hence every $X_2$ of $D_{P_{n-1}^{-1}}$, $i = 0$ belong to $D_{P_{n}^{1}}$, $i$ by addition of $\{n\}$.

Let $X_2$ be the total dominating set of $P_{n-2}^{1}$ with cardinality $i$. When $n = 2$, $n = 3$ or $n = 4$. When $n = 2$ or $n = 3$, then adjoin $n - 2$ with $X_2$. Hence every $X_2$ of $D_{P_{n-2}^{1}}$, $i = 0$ belong to $D_{P_{n}^{1}}$, $i$ by addition of $\{n\}$.

Let $X_2$ be the total dominating set of $P_{n-3}^{1}$ with cardinality $i$. When $n = 2$, $n = 3$ or $n = 4$. When $n = 2$ or $n = 3$, then adjoin $n - 2$ with $X_2$. Hence every $X_2$ of $D_{P_{n-2}^{1}}$, $i = 0$ belong to $D_{P_{n}^{1}}$, $i$ by addition of $\{n\}$.

Let $X_2$ be the total dominating set of $P_{n-4}^{1}$ with cardinality $i$. When $n = 2$, $n = 3$ or $n = 4$. When $n = 2$ or $n = 3$, then adjoin $n - 2$ with $X_2$. Hence every $X_2$ of $D_{P_{n-2}^{1}}$, $i = 0$ belong to $D_{P_{n}^{1}}$, $i$ by addition of $\{n\}$.

Let $X_2$ be the total dominating set of $P_{n-4}^{1}$ with cardinality $i$. When $n = 2$, $n = 3$ or $n = 4$. When $n = 2$ or $n = 3$, then adjoin $n - 2$ with $X_2$. Hence every $X_2$ of $D_{P_{n-2}^{1}}$, $i = 0$ belong to $D_{P_{n}^{1}}$, $i$ by addition of $\{n\}$.

Let $X_2$ be the total dominating set of $P_{n-4}^{1}$ with cardinality $i$. When $n = 2$, $n = 3$ or $n = 4$. When $n = 2$ or $n = 3$, then adjoin $n - 2$ with $X_2$. Hence every $X_2$ of $D_{P_{n-2}^{1}}$, $i = 0$ belong to $D_{P_{n}^{1}}$, $i$ by addition of $\{n\}$.

Let $X_2$ be the total dominating set of $P_{n-4}^{1}$ with cardinality $i$. When $n = 2$, $n = 3$ or $n = 4$. When $n = 2$ or $n = 3$, then adjoin $n - 2$ with $X_2$. Hence every $X_2$ of $D_{P_{n-2}^{1}}$, $i = 0$ belong to $D_{P_{n}^{1}}$, $i$ by addition of $\{n\}$.

Let $X_2$ be the total dominating set of $P_{n-4}^{1}$ with cardinality $i$. When $n = 2$, $n = 3$ or $n = 4$. When $n = 2$ or $n = 3$, then adjoin $n - 2$ with $X_2$. Hence every $X_2$ of $D_{P_{n-2}^{1}}$, $i = 0$ belong to $D_{P_{n}^{1}}$, $i$ by addition of $\{n\}$.

Let $X_2$ be the total dominating set of $P_{n-4}^{1}$ with cardinality $i$. When $n = 2$, $n = 3$ or $n = 4$. When $n = 2$ or $n = 3$, then adjoin $n - 2$ with $X_2$. Hence every $X_2$ of $D_{P_{n-2}^{1}}$, $i = 0$ belong to $D_{P_{n}^{1}}$, $i$ by addition of $\{n\}$.

Let $X_2$ be the total dominating set of $P_{n-4}^{1}$ with cardinality $i$. When $n = 2$, $n = 3$ or $n = 4$. When $n = 2$ or $n = 3$, then adjoin $n - 2$ with $X_2$. Hence every $X_2$ of $D_{P_{n-2}^{1}}$, $i = 0$ belong to $D_{P_{n}^{1}}$, $i$ by addition of $\{n\}$.
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If \( n \geq 2 \) or \( n \geq 3 \) in \( X \), where \( \mathcal{D}(P_n, i) \) and \( \mathcal{D}(P_{n-1}, i) \), then \( d_i(P_n, i) = d_i(P_{n-1}, i) + d_i(P_{n-2}, i) \) and \( d_i(P_{n-3}, i) + d_i(P_{n-4}, i) \)

**Theorem 3.1**

If \( D_i(P_n^2, i) \) is the family of the total dominating sets of \( P_n^2 \) with cardinality \( i \) where \( i > \left\lceil \frac{n}{2} \right\rceil + 1 \), then \( d_i(P_n^2, i) = d_i(P_{n-1}, i) + d_i(P_{n-2}, i) + d_i(P_{n-3}, i) + d_i(P_{n-4}, i) \)

**Proof:**

From theorem (4.7) and (4.8), we consider all the three cases as given below, Where \( i > \left\lceil \frac{n}{2} \right\rceil + 1 \),

(i) If \( D_i(P_n^2, i) = \emptyset \), \( D_i(P_{n-1}, i) = \emptyset \), \( D_i(P_{n-2}, i) = \emptyset \), and \( D_i(P_{n-3}, i) = \emptyset \) then \( D_i(P_n^2, i) = \{[n]\} \)

(ii) If \( D_i(P_n^2, i) = \emptyset \), \( D_i(P_{n-1}, i) = \emptyset \), \( D_i(P_{n-2}, i) = \emptyset \), and \( D_i(P_{n-3}, i) = \emptyset \) then \( D_i(P_n^2, i) = \{[n]-[x] \in \{n\}\} \)

(iii) If \( D_i(P_n^2, i) = \emptyset \), \( D_i(P_{n-1}, i) = \emptyset \), \( D_i(P_{n-2}, i) = \emptyset \), \( D_i(P_{n-3}, i) = \emptyset \) then \( D_i(P_n^2, i) = \{(X, n) \in X \} \)

Using theorem (1.9), we obtain \( d_i(P_n^2, i) \) for \( I \leq 15 \) as shown in Table 1.1.

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<td>1179</td>
<td>427</td>
<td>103</td>
<td>15</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>42</td>
<td>411</td>
<td>1419</td>
<td>2600</td>
<td>2956</td>
<td>2247</td>
<td>1179</td>
<td>427</td>
<td>103</td>
<td>15</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

By theorem 3.1, we obtain \( d_i(P_n^2, i) \), \( n \geq 2 \) for \( 2 \leq n \leq 15 \) as shown in Table 1.1. There are interesting relationship between numbers in this Table. In the following theorem we obtain some properties of \( d_i(P_n^2, i) \).

**Theorem 3.2**

Let \( P_n^2 \), \( n \geq 2 \) be the square of path with \( |V(P_n^2)| = n \). Then the following properties hold for the coefficients of \( D_i(P_n^2, i) \):

(i) For \( n \geq 2 \), \( d_i(P_n^2, n) = 1 \)

(ii) For \( n \geq 3 \), \( d_i(P_n^2, n-1) = n \)

(iii) For \( n \geq 5 \), \( d_i(P_n^2, n-2) = nc_2 \)
(iv) For \( n \geq 7 \), \( d_t(P_n^2,n-3) = nc_1-2(n-1) \)

(v) For \( n \geq 8 \), \( d_t(P_n^2,n-4) = nc_1- (n^2-2n-9) \)

(vi) For \( k \geq 1 \), \( d_t(P_{3k-2},2k) = 1 \)

(vii) For \( k \geq 1 \), \( d_t(P_{3k-2},k+1) = k+2 \)

**Proof of (i)**

Since for any graph \( G \) with \( n \) vertices, \( d_t(G,n) = 1 \), then \( d_t(P_n^2,n) = 1 \).

**Proof of (ii)**

To prove \( d_t(P_n^2,n-1) = n \), for \( n \geq 3 \). Since \( D_t(P_n^2,n-1) = \{ [n]-\{x\}/ x \in [n], | D_t(P_n^2,n-1) | = nc_1 = n \)

Therefore \( d_t(P_n^2,n-1) = n \).

**Proof of (iii)**

To prove \( d_t(P_n^2,n-2) = nc_2 - 2 \), for \( n \geq 5 \)

We apply induction on \( n \). When \( n = 5 \)

L.H.S = \( d_t(P_5^2,3) = 8 \) (from table), R.H.S = \( 5c_2 - 2 = 8 \)

Therefore the result is true for \( n = 5 \).

Suppose that the result is true for all natural numbers less than \( n \), and we prove it for \( n \).

We have, \( d_t(P_n^2,n-2) = d_t(P_{n-1}^2,n-3) + d_t(P_{n-2}^2,n-3) + d_t(P_{n-3}^2,n-3) + d_t(P_{n-4}^2,n-3) \)

\[ \frac{1}{2} (n-1) (n-2) - 2 + n - 1 = \frac{1}{2} (n^2 - 4) = \frac{1}{2} [n(n-1)] - 2 = nc_2 - 2 \), for \( n \geq 5 \)

Hence the result is true for all \( n \).

Hence by induction hypothesis, we have

\( d_t(P_n^2,n-2) = nc_2 - 2 \), for \( n \geq 5 \)

**Proof of (iv)**

To prove \( d_t(P_n^2,n-3) = \frac{1}{6} [n(n-1)(n-2)] - 2(n-1) \), for every \( n \geq 7 \)

We apply induction on \( n \).

When \( n = 7 \), L.H.S = \( d_t(P_7^2,4) = 23 \) (from table), R.H.S = \( \frac{1}{6} [7(7-1)(7-2)] - 2(7-1) = 23 \)

Therefore the result is true for \( n = 7 \).

Suppose that the result is true for all natural numbers less than \( n \), and we prove it for \( n \).

We have, \( d_t(P_n^2,n-3) = d_t(P_{n-1}^2,n-4) + d_t(P_{n-2}^2,n-4) + d_t(P_{n-3}^2,n-4) + d_t(P_{n-4}^2,n-4) \)

\[ \frac{1}{6} (n-1) (n-2) (n-3) - 2(n-2) + \frac{1}{2} [(n-2)(n-3)] - 2 + n - 3 + 1 \]

\[ = \frac{1}{6} (n-1) (n^2 - 5n + 6) - 2n + 4 + \frac{1}{2} (n^2 - 5n + 6) - 2 + n - 3 + 1 \]

\[ = \frac{1}{6} (n^3 - 5n^2 + 6n + 6n^2 - 6n + 3n^2 - 15n + 18) = \frac{1}{6} (n^3 - 3n^2 + 10n + 12) \]

\[ = \frac{1}{6} (n^3 - 3n^2 + 2n - 12 + 12) = \frac{1}{6} (n^3 - 3n^2 + 2n) - \frac{1}{6} (12n - 12) = \frac{1}{6} n(n-1)(n-2) - 2(n-1) \) for \( n \geq 7 \)

Therefore the result is true for all \( n \). Hence by induction hypothesis, we have

\( d_t(P_n^2,n-3) = \frac{1}{6} [n(n-1)(n-2)] - 2(n-1), \) for every \( n \geq 7 \).

**Proof of (v)**

To prove \( d_t(P_n^2,n-4) = nc_2 -(n^2-2n-9) \), for \( n \geq 8 \).

We apply induction on \( n \).

When \( n = 8 \), L.H.S = \( d_t(P_8^2,4) = 31 \) (from table), R.H.S = \( \frac{1}{24} [8(8-1)(8-2)(8-4)] - 64 - 16 - 9 \) = 31

Therefore the result is true for \( n = 8 \).

Suppose that the result is true for all natural numbers less than \( n \), and we prove it for \( n \).

We have, \( d_t(P_n^2,n-4) = d_t(P_{n-1}^2,n-5) + d_t(P_{n-2}^2,n-5) + d_t(P_{n-3}^2,n-5) + d_t(P_{n-4}^2,n-5) \)

\[ = \frac{1}{24} (n-1)(n-2)(n-3)(n-4) + \frac{1}{24} (n-2)(n-3)(n-4) + \frac{1}{24} (n-3)(n-4) + \frac{1}{24} (n^2 + 2n - 1 - 2n - 1) + \frac{1}{24} + \frac{1}{24} (n^2 - 2n + 12) + \frac{1}{24} (n^2 - 2n + 12) + \frac{1}{24} (n^2 - 3n + 6) \]

\[ = \frac{1}{24} (n^3 - 7n^2 + 12n - 3n + 6) + \frac{1}{24} (n^3 - 7n^2 + 12n + 12) + \frac{1}{24} (n^3 - 7n^2 + 12n + 12) + \frac{1}{24} (n^2 - 3n + 6) \]

\[ = \frac{1}{24} (n^3 - 7n^2 + 12n^2 - 3n + 6) + \frac{1}{24} (n^3 - 2n^2 + 3n + 6) + \frac{1}{24} (n^3 - 2n^2 + 3n + 6) + \frac{1}{24} (n^2 - 3n + 6) \]

\[ = \frac{1}{24} (n^3 - 7n^2 + 12n^2 - 3n + 6) + \frac{1}{24} (n^3 - 2n^2 + 3n + 6) + \frac{1}{24} (n^3 - 2n^2 + 3n + 6) + \frac{1}{24} (n^2 - 3n + 6) \]
Total Dominating Sets And Total Domination Polynomials Of Square Of Paths

\[
\frac{1}{24}[n^4-6n^3-13n^2+42n+216]=\frac{1}{24}[n^4-6n^3+11n^2-24n^2-6n+48n+216]
\]

\[
=\frac{1}{24}[n^4-6n^3-11n^2-6n]+\frac{1}{24}[n^4-6n^3+48n+216]=\frac{1}{24}[n(n-1)(n-2)(n-3)-(n^2-2n-9)]
\]

Therefore the result is true for all \( n \).

Hence by induction hypothesis, we have \( d_t(P_n^2,n-4) = nc_{n-1}(n^2-2n-9) \), for \( n \geq 8 \).

**Proof of (vi)**

To prove \( d_t(P_n^2,k) = 1 \), for \( k \geq 1 \).

Since, \( d_t(P_7^2,2) = \{3,5, \ldots, 7k-4, 7k-5\} \).

Therefore, \( d_t(P_n^2,2k) = D_t(P_n^2,2k) = 1 \).

**Proof of (vii)**

To prove \( d_t(P_{3k-2},k+1) = k+2 \), for \( k \geq 1 \)

When \( K = 1, D_t(P_{3k}^2,2) = \{1,2\} \). Therefore, \( d_t(P_3^2,2) = 3 \)

Assume that the result is true for all natural numbers \( m \) less than \( k \).

Therefore, \( d_t(P_{3m-2},m+1) = m+2 \), for \( m < k \),

\( d_t(P_{3m+1},m+2) = m+3 \)

\( d_t(P_{3(k+1)-2},m+2) = (m+1)+2, d_t(P_{3k-2},k+1) = k+2 \).

II. Conclusion:

We obtain total domination sets and total domination polynomial square of paths. Similarly we can find total domination sets and total domination polynomial of specified graphs.

**References**


