Generalised \( CR \)-submanifolds of an \((\varepsilon, \delta)\)-trans-Sasakian manifold with certain connection

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Abstract: In this paper, generalised \( CR \)-submanifolds of an \((\varepsilon, \delta)\)-trans-Sasakian manifolds with semi-symmetric non-metric connection are studied. Moreover, integrability conditions of the distributions on generalised \( CR \)-submanifolds of an \((\varepsilon, \delta)\)-trans-Sasakian manifolds with semi-symmetric non-metric connection and geometry of leaves with semi-symmetric non-metric connection have been discussed.

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I. Introduction

I.1. Introduction

Ion Mihai [1] introduced a new class of submanifolds called Generalised \( CR \)-submanifolds of Kaehler manifolds and also studied generalised \( CR \)-submanifolds of Sasakian manifolds [2]. In 1985, Oubina [3] introduced a new class of almost contact Riemannian manifolds known as trans-Sasakian manifolds. After M. H. Shahid studied \( CR \)-submanifolds of trans-Sasakian manifold [4] and generic submanifolds of trans-Sasakian manifold [5]. In 2001, A. Kumar and U.C. De [6] studied generalised \( CR \)-submanifolds of a trans-Sasakian manifolds. In 1993, A. Bejancu and K. L. Duggal [7] introduced the concept of \((\varepsilon)\)-Sasakian manifolds. Then U. C. De and A. Sarkar [8] introduced \((\varepsilon)\)-Kenmotsu manifolds. The existence of a new structure on indefinite metrics has been discussed. Moreover, Bhattacharyya [9] studied the contact \( CR \)-submanifolds of indefinite trans-Sasakian manifolds. Recently, Nagaraja et. al. [10] introduced the concept of \((\varepsilon, \delta)\)-trans-Sasakian manifolds which generalised the notion of \((\varepsilon)\)-Sasakian as well as \((\delta)\)-Kenmotsu manifolds. In 2010, Cihan Özgür [11] studied the submanifolds of Riemannian manifold with semi-symmetric non-metric connection. Moreover, Özgür and others also studied the different structures with semi-symmetric non-metric connection in ([12], [13]). On other hand, some properties of semi-invariant submanifolds, hypersurfaces and submanifolds with semi-symmetric non-metric connection were studied in ([14], [15], [16]). Thus motivated sufficiently from the above studies, in this paper we study generalised \( CR \)-submanifolds of an \((\varepsilon, \delta)\)-trans-Sasakian manifolds with semi-symmetric non-metric connection.

We know that a connection \( \nabla \) with a Riemannian metric \( g \) on a manifold \( M \) is called metric such that \( \nabla g = 0 \), otherwise it is non-metric. Further it is said to be a semi-symmetric linear connection [17]. A linear connection \( \nabla \) is said to be a semi-symmetric connection if its torsion tensor is of the form

\[
T(X, Y) = \eta(Y)X - \eta(X)Y,
\]

where \( \eta \) is a 1-form. A study of semi-symmetric connection on Riemannian manifold was enriched by K. Yano [18]. In 1992, Agashe and Chaffle [19] introduced the notion of semi-symmetric non-metric connection. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection.

Semi-Symmetric metric connection plays an important role in the study of Riemannian manifolds, there are various physical problems involving the semi-symmetric metric connection. For example if a man is moving on the surface of the earth always facing one definite point, say Mekka or Jerusalem or the North pole, then this displacement is semi-symmetric and metric [20].

In this paper, we study Generalised \( CR \)-submanifolds of an \((\varepsilon, \delta)\)-trans-Sasakian manifolds with a semi-symmetric non-metric connection. This paper is organized as follows. In section 2, we give a brief introduction of generalised \( CR \)-submanifolds of an \((\varepsilon, \delta)\)-trans-Sasakian manifold and give an example. In
section 3, we discuss some Basic Lemmas. In section 4, integrability of some distributions discuss. In section 5, Geometry of leaves of Generalised CR -submanifolds of an \((\epsilon, \delta)\)-trans-Sasakian manifold with semi-symmetric non-metric connection have been discussed.

II. \((\epsilon, \delta)\)-trans-Sasakian manifolds

Let \(\widetilde{M}\) be an almost contact metric manifold with an almost contact metric structure \((\phi, \xi, \eta, g)\), where \(\phi\) is a \((1,1)\) tensor field, \(\xi\) is a vector field, \(\eta\) is a 1-form and \(g\) is a compatible Riemannian metric such that

\[
\phi^2 = -I + \eta \otimes \xi, \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1, \quad (2.1)
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \beta g(\phi X, Y) - \alpha g(X, \phi Y), \quad (2.2)
\]

\[
g(\xi, \xi) = \epsilon \quad (2.3)
\]

\[
g(X, \phi Y) = -g(\phi X, Y), \quad \epsilon g(\xi, \xi) = \eta(X) \quad (2.4)
\]

for all vector fields \(X, Y\) on \(\mathbb{T}\mathcal{M}\), where \(\epsilon = g(\xi, \xi) = \pm 1\). An \((\epsilon)\)-almost contact metric manifold is called an \((\epsilon, \delta)\)-trans-Sasakian manifold \(\text{[10]}\) if

\[
\nabla_X \phi(Y) = \alpha(g(X, Y) \xi - \epsilon \eta(Y) X) + \beta(g(\phi X, Y) \xi - \delta \eta(Y) \phi X) \quad (2.5)
\]

for some smooth functions \(\alpha\) and \(\beta\) on \(\mathcal{M}\) and \(\epsilon = \pm 1, \delta = \pm 1\). For \(\beta = 0, \alpha = 1\), an \((\epsilon, \delta)\)-trans-Sasakian manifolds reduces to \((\epsilon)\)-Sasakian and for \(\alpha = 0, \beta = 1\) it reduces to a \((\delta)\)-Kenmotsu manifolds. From (2.5) it follows that

\[
\nabla_X \xi = -\epsilon \alpha \phi X - \beta \delta \phi^2 X. \quad (2.6)
\]

for any vector field \(X\) tangent to \(\widetilde{M}\).

Example of \((\epsilon, \delta)\)-trans-Sasakian manifolds

Consider the three dimensional manifold \(M = \{(x, y, z) \in \mathbb{R}^3 | z \neq 0\}\), where \((x, y, z)\) are the cartesian coordinates in \(\mathbb{R}^3\) and let the vector fields are

\[
e_1 = \frac{e^x}{z} \frac{\partial}{\partial x}, \quad e_2 = \frac{e^y}{z} \frac{\partial}{\partial y}, \quad e_3 = \frac{-2(\epsilon + \delta)}{z} \frac{\partial}{\partial z},
\]

where \(e_1, e_2, e_3\) are linearly independent at each point of \(M\). Let \(g\) be the Riemannian metric defined by

\[
g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = \epsilon, \quad g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0,
\]

where \(\epsilon = \pm 1\). Let \(\eta\) be the 1-form defined by \(\eta(X) = \epsilon g(X, \xi)\) for any vector field \(X\) on \(M\), let \(\phi\) be the \((1,1)\) tensor field defined by \(\phi(e_1) = e_2, \phi(e_2) = e_3, \phi(e_3) = 0\).

Then by using the linearity of \(\phi\) and \(g\), we have \(\phi^2 X = -X + \eta(X) \xi\), with \(\xi = e_3\).

Further \(g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X) \eta(Y)\) for any vector fields \(X\) and \(Y\) on \(M\). Hence for \(e_3 = \xi\), the structure defines an \((\epsilon)\)-almost contact structure in \(\mathbb{R}^3\).

Let \(\nabla\) be the Levi-Civita connection with respect to the metric \(g\), then we have

\[
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y])
\]

which is know as Koszul’s formula.

We, also have

\[
\nabla_{e_1} e_3 = -\frac{(\epsilon + \delta)}{z} e_1, \quad \nabla_{e_2} e_3 = -\frac{(\epsilon + \delta)}{z} e_2, \quad \nabla_{e_1} e_2 = 0,
\]
using the above relation, for any vector $X$ on $M$, we have
$$\nabla_X \xi = -\alpha \xi X - \beta \delta \xi^2 X, \quad \text{where} \quad \alpha = \frac{1}{\varepsilon} \quad \text{and} \quad \beta = -\frac{1}{\varepsilon}.$$\[180pt]
Hence $(\phi, \xi, \eta, g)$ structure defines the $(\varepsilon, \delta)$-trans-Sasakian structure in $\mathbb{R}^3$.

III. Semi-symmetric non-metric connection

We remark that owing to the existence of the 1-form $\eta$, we can define a semi-symmetric non-metric connection $\nabla$ in almost contact metric manifold by
$$\nabla_X Y = \nabla_X Y + \eta(Y) X, \quad (3.1)$$
where $\nabla$ is the Riemannian connection with respect to $g$ on $n$-dimensional Riemannian manifold and $\eta$ is a 1-form associated with the vector field $\xi$ on $M$ defined by
$$\eta(X) = g(X, \xi). \quad (3.2)$$
[19] By (3.1) the torsion tensor $T$ of the connection $\nabla$ is given by
$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \quad (3.3)$$
Also, we have
$$T(X, Y) = \eta(Y) X - \eta(X) Y. \quad (3.4)$$
A linear connection $\nabla$, satisfying (3.4) is called a semi-symmetric connection. $\nabla$ is called a metric connection if
$$\nabla g = 0$$
otherwise, if $\nabla g \neq 0$, then $\nabla$ is said to be non-metric connection. Furthermore, from (3.1), it is easy to see that
$$\nabla_X g(Y, Z) = (\nabla_X g)(Y, Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$
which implies
$$\nabla_X g(Y, Z) = \eta(Y) g(X, Z) - \eta(Z) g(X, Y) \quad (3.5)$$
for all vector fields $X$, $Y$, $Z$ on $M$. Therefore in view of (3.4) and (3.5) $\nabla$ is a semi-symmetric non-metric connection.

for all $X, Y \in TM$. Now from (3.1), (2.5) and (2.6), we have
$$\nabla_X \phi Y = \alpha \{ (g(X, Y) \xi - \varepsilon \eta(Y) X \} + \beta (g(\phi X, Y)) \xi + (1 - \beta \delta) \eta(Y) \phi X.$$
(3.6)
From (3.6) it follows that
$$\nabla_X \xi = X - \alpha \xi X - \beta \delta \xi^2 X \quad (3.7)$$
for any vector field $X$ tangent to $M$.

Now, let $M$ be a submanifold isometrically immersed in an $(\varepsilon, \delta)$-trans-Sasakian manifold $\bar{M}$ such that the structure vector field $\xi$ of $\bar{M}$ is tangent to submanifolds $M$. We denote by $\{ \}$ is the 1-dimensional distribution spanned by $\xi$ on $M$ and by $\{ \xi^1 \}$ the complementary orthogonal distribution to $\xi$ in $TM$.

For any $X \in TM$, we have $g(\phi X, \xi) = 0$. Then we have
$$\phi X = BX + CX, \quad (3.8)$$
where $BX \in \{ \xi^1 \}$ and $CX \in T^+M$. Thus $X \rightarrow BX$ is an endomorphism of the tangent bundle $TM$ and $X \rightarrow CX$ is a normal bundle valued 1-form on $M$. 

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Definition. A submanifold of \( M \) of an almost contact metric manifolds \( \overline{M} \) with an \((\varepsilon, \delta)\)-trans-Sasakian metric structure \((\phi, \xi, \eta, g)\) is said to be a generalised CR-submanifold if

\[
D^\perp_x = T_x M \cap \phi T_x M; \quad \text{for} \quad x \in M
\]
defines a differentiable sub-bundle of \( T_x M \). Thus for \( X \in D^\perp \) one has \( BX = 0 \).

We denote by \( D \) the complementary orthogonal sub-bundle to \( D^\perp \oplus \{\xi\} \) in \( TM \).

For any \( X \in D \), \( BX \neq 0 \). Also we have \( BD = D \).

Thus for a generalised CR-submanifold \( M \), we have the orthogonal decomposition

\[
TM = D \oplus D^\perp \oplus \{\xi\}. \quad (3.9)
\]

IV. Basic Lemmas

Let \( M \) be a generalised CR-submanifold of an \((\varepsilon, \delta)\)-trans-Sasakian manifold \( \overline{M} \). We denote by \( g \) both Riemannian metrics on \( \overline{M} \) and \( M \).

For each \( X \in TM \), we can write

\[
X = PX + QX + \eta(X)\xi, \quad (4.1)
\]

where \( PX \) and \( QX \) belong to the distribution \( D \) and \( D^\perp \) respectively.

For any \( N \in T_x^\perp M \), we can write

\[
\phi X = tN + fN, \quad (4.2)
\]

where \( tN \) is the tangential part of \( \phi N \) and \( fN \) is the normal part of \( \phi N \).

By using (2.2) we have

\[
g(\phi X, CY) = g(\phi X, BY + CY) = g(\phi X, \phi Y) = g(X, Y) = 0,
\]

for \( X \in D^\perp_x \) and \( Y \in D_x \). Therefore

\[
g(\phi D^\perp, CD) = 0. \quad (4.3)
\]

We denote by \( \nu \) the orthogonal complementray vector bundle to \( \phi D^\perp \oplus CD \) in \( T^\perp M \).

Thus, we have

\[
T^\perp M = \phi D^\perp \oplus CD \oplus \nu \quad (4.4)
\]

Lemma 4.1. The morphism \( t \) and \( f \) satisfy

\[
t(\phi D^\perp) = D^\perp \quad (4.5)
\]

\[
t(CD) \subset D \quad (4.6)
\]

Proof. For \( X \in D^\perp \) and \( Y \in D \),

\[
g(t\phi, Y) = g(t\phi X + f\phi Y, Y) = g(\phi^2 X, Y) = -g(X, Y) = 0
\]

\[
g(t\phi X, \xi) = g(\phi^2 X, \xi) = -g(\phi X, \phi \xi) = 0.
\]

Therefore, \( t(\phi D^\perp) \subset D^\perp \).

For \( X \in D^\perp \), we have

\[-X = \phi^2 X = t\phi X + f\phi X, \quad \text{which implies} \quad -X = t\phi X.\]

Consequently, \( D^\perp \subset t(\phi D^\perp) \). Hence the equation (4.5) follows. The equation (4.6) is trivial.

Let \( M \) be a submanifold of a Riemannian manifold \( \overline{M} \) with Riemannian metric \( g \). Then Gauss and Weingarten formulae are given respectively by
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\[
\nabla X Y = \nabla X Y + h(X, Y) \quad (X, Y \in TM),
\]

\[
\nabla X N = -A_N X + \nabla X N + \eta(N) X \quad (N \in T^\perp M),
\]

where \(\nabla\), \(\nabla\) and \(\nabla^\perp\) respectively the semi-symmetric non-metric, induced connection and induced normal connections in \(\overline{M}\), \(M\) and the normal bundle \(T^\perp M\) of \(M\) respectively and \(h\) is the second fundamental form related to \(A\) by

\[
g(h(X, Y), N) = g(A_N X, Y)
\]

for \(X, Y \in TM\) and \(N \in T^\perp M\).

We denote

\[
u(X, Y) = \nabla_{X} BPY - A_{CPY} X - A_{\phi Y} X.
\]

**Lemma 4.2.** Let \(M\) be a generalised \(CR\)-submanifold of an \((\epsilon, \delta)\)-trans-Sasakian manifold \(\overline{M}\) with semi-symmetric non-metric connection. Then we have

\[
P(u(X, Y)) - BP\nabla X Y - Pth(X, Y) = -\alpha \xi \eta(Y) PX
\]

\[
-(1 - \beta \delta) \eta(Y) PBX - 2\eta(CPY) PX,
\]

\[
Q(u(X, Y)) - Qth(X, Y) = -\alpha \xi \eta(Y) QX - (1 - \beta \delta) \eta(Y) QBX
\]

\[
-2\eta(CPY) QX,
\]

\[
\eta(u(X, Y)) = \alpha g(\phi X, \phi Y) + \beta g(\phi B X, \phi Y) - 2\eta(CPY) \eta(X) \xi,
\]

\[
h(X, BPY) + \nabla_{X} CPY + \phi QY - CP \nabla_{X} Y - \phi Q \nabla_{X} Y - fh(X, Y)
\]

\[
= (1 - \beta \delta) \eta(Y) CX,
\]

for \(X, Y \in TM\).

**Proof.** For \(X, Y \in TM\) by using \((3.8), (4.1), (4.2), (4.7), (4.8)\) in \((3.6)\), we have

\[
\nabla_{X} BPY + h(X, BPY) - A_{CPY} X + \nabla_{X} CPY + \eta(CPY) X - A_{\phi Y} X + \nabla_{X} \phi QY
\]

\[
- BP \nabla_{X} Y - CP \nabla_{X} Y - \phi Q \nabla_{X} Y - Pth(X, Y) - Qth(X, Y) - fh(X, Y)
\]

\[
= \alpha \{g(X, Y) \xi - \alpha \eta(Y) X\} + \beta g(\phi X, Y) \xi + (1 - \beta \delta) \eta(Y) \phi X.
\]

Then \((4.11), (4.12), (4.13)\) and \((4.14)\) are obtaining by taking the components of each vector bundles \(D\), \(D^\perp\), \(\{\xi\}\) and \(T^\perp(M)\) respectively.

**Lemma 4.3.** Let \(M\) be a generalised \(CR\)-submanifold of an \((\epsilon, \delta)\)-trans-Sasakian manifold \(\overline{M}\) with semi-symmetric non-metric connection. Then we have

\[
P(t \nabla_{X} N + A_{N} X - \nabla_{X} t N) = BPA_{N} X - \eta(f N) PX,
\]

\[
Q(t \nabla_{X} N + A_{N} X - \nabla_{X} t N) = -\eta(f N) QX,
\]

\[
\eta(A_{N} X - \nabla_{X} t N) = -\beta g(CX, N) + \eta(f N) \eta(X) \xi,
\]

\[
h(X, t N) + \phi QA_{N} X + \nabla_{X} f N + CPA_{N} X = f \nabla_{X} N
\]

for \(X \in TM\) and \(N \in T^\perp M\).
Proof. For $X \in TM$ and $N \in T^\bot M$ by using the equations (3.8), (4.1), (4.2), (4.7) and (4.8) in (3.6), we get

$$P\nabla_X tN + Q\nabla_X tN + \eta(\nabla_X tN) + h(X, tN) - PA_{JN} X - \eta(fN)PX - QA_{JN} X$$

$$-\eta(fN)QX - \eta(A_{JN}X) + \nabla_X fN + \eta(fN)\eta(X)\xi + BPA_X X + CPA_{JN} X$$

$$+ \phi QA_{JN} X - Pt\nabla_X N - Q\nabla_X N - f\nabla_X N = \beta g(CX, N)$$

Then (4.15), (4.16), (4.17) and (4.18) are obtaining by taking the components of each vector bundles $D$, $D^\bot$, $\{\xi\}$ and $T^\bot (M)$ respectively.

**Lemma 4.4.** Let $M$ be a generalised $CR$-submanifold of an $(\epsilon, \delta)$-trans-Sasakian manifold $\bar{M}$ with semi-symmetric non-metric connection. Then we have

$$\nabla_X \xi = PX + \beta \delta X - \alpha BX,$$

for $X \in D$ \hspace{1cm} (4.19)

$$h(X, \xi) = QX - \alpha CX \text{ and } (1 - \beta \delta)\eta(X) = 0,$$

for $X \in D$ \hspace{1cm} (4.20)

$$\nabla_Y \xi = PY + \beta \delta Y,$$

for $Y \in D^\bot$ \hspace{1cm} (4.21)

$$h(Y, \xi) = QY - \alpha \phi Y; \quad \eta(Y)(1 - \beta \delta) = 0,$$ for $Y \in D^\bot$ \hspace{1cm} (4.22)

$$\nabla_\xi \xi = P\xi$$ \hspace{1cm} (4.23)

$$h(\xi, \xi) = Q\xi; \quad \beta \delta = 1.$$ \hspace{1cm} (4.24)

**Proof.** The proof of above lemma from (3.7) by using (3.8), (4.1) and (4.7).

**Lemma 4.5** Let $M$ be a generalised $CR$-submanifold of an $(\epsilon, \delta)$-trans-Sasakian manifold $\bar{M}$ with semi-symmetric non-metric connection. Then we have

$$A_{\phi \xi} Y = A_{\phi \eta} X,$$

for $X, Y \in D^\bot$. \hspace{1cm} (4.25)

**Proof.** By using (2.2), (2.3), (4.7) and (4.9), we get

$$g(A_{\phi \xi} Y, Z) = g(h(Y, Z), \phi X) = g(\nabla_X Y, \phi X) = -g(\phi \nabla_X Y, X)$$

$$= -g(\nabla_{\phi \xi} Y, X) = g(\phi Y, \nabla_X Z) = g(h(Z, X), \phi Y) = g(h(X, Z), \phi Y)$$

$$= g(A_{\phi \xi} Y, Z),$$

for $X, Y \in D^\bot$ and $Z \in TM$. Hence the Lemma follows.

**Lemma 4.6.** Let $M$ be a generalised $CR$-submanifold of an $(\epsilon, \delta)$-trans-Sasakian manifold $\bar{M}$ with semi-symmetric non-metric connection. Then we have

$$\nabla_\xi V \in D^\bot, \text{ for } V \in D^\bot \text{ and } (4.26)$$

$$\nabla_\xi W \in D, \text{ for } W \in D.$$ \hspace{1cm} (4.27)

**Proof.** Let us take $X = \xi$ and $V = \phi N$ in (4.15), where $N \in \phi D$. Taking account that $tN = \phi N$ and $fN = 0$ we get

$$P\nabla_\xi V = Pr\nabla_\xi N - BPA_X \xi.$$ \hspace{1cm} (4.28)

The first relation of (4.20) gives

$$g(\phi \nabla_X \xi, W) = g(A_{\phi \xi} W, X) = g(h(W, \xi, X)) = -\alpha \alpha g(CW, N) + g(QW, N) = 0$$

for $W \in D$. Hence, (4.28) becomes
\[ P\nabla_{\xi} V = P\nabla_{\xi} N. \quad (4.29) \]

On the other hand (4.18) implies
\[ h(\xi, V) = f\nabla_{\xi} N - \phi QA_N \xi. \quad (4.30) \]

For \( V \in D^\perp \), \( h(\xi, V) = h(V, \xi) = -\alpha \phi V \in \phi D^\perp \), by (3.22)

Now for \( X \in D^\perp \) by using the lemma (4.5) and of (4.9), we have
\[ g(h(\xi, V), \phi X) = g(h(V, \xi), \phi X) = g(A_{\phi^r} V, \xi) = g(A_{\phi^r} X, \xi) \]
\[ = g(h(X, \xi), \phi V) = g(h(X, \xi), -N) = -g(A_N \xi, X) = -g(\phi A_N \xi, \phi X) \]
\[ = -g(\phi PA_N \xi, \phi X) - g(\phi QA_N \xi, \phi X) = -g(\phi QA_N \xi, \phi X) \]

since \( CD^\perp \phi = \phi D^\perp \).

Therefore, \( h(\xi, V) = -\phi QA_N \xi \), which together with (4.30) implies \( f\nabla_{\xi} N = 0 \).

Hence \( \nabla_{\xi} N = \phi D^\perp \), since \( f \) is an automorphism of \( CD \oplus V \). Thus, \( t\nabla_{\xi} N = D^\perp \) and from (4.29) it follows that
\[ P\nabla_{\xi} V = 0, \quad \text{for all} \ V \in D^\perp \quad (4.31) \]

Next from (4.17), we have
\[ \eta(\nabla_{\xi} V) = 0 \quad (4.32) \]

for all \( V = \phi D \in D^\perp \), where \( N \in \phi D^\perp \). Hence (4.26) follows from (4.31) and (4.32).

Finally using the (4.1), (4.23) and (4.26), we have
\[ g(\nabla_{\xi} W, X) = g(\nabla_{\xi} W, PX) \]

for \( X = TM \) and \( W \in D \). Thus we have \( \nabla_{\xi} W \in D \), for \( W \in D \) and this completes the proof.

**Corollary 4.1.** Let \( M \) be a generalised \( CR \)-submanifold of an \((\epsilon, \delta)\)-trans-Sasakian manifold \( \overline{M} \) with semi-symmetric non-metric connection. Then we have
\[ [Y, \xi] \in D^\perp, \quad \text{for} \ Y \in D^\perp \quad (4.33) \]
\[ [X, \xi] \in D, \quad \text{for} \ X \in D \quad (4.34) \]

The above corollary follows immediate consequences of the Lemma (4.4) and Lemma (4.6).

### V. Integrability of Distributions

**Theorem 5.1.** Let \( M \) be a generalised \( CR \)-submanifold of an \((\epsilon, \delta)\)-trans-Sasakian manifold \( \overline{M} \) with semi-symmetric non-metric connection. Then the distribution \( D^\perp \) is always involutive if and only if
\[ g([X, Y], \xi) - 2\beta \delta g(X, Y) = 0. \quad (5.1) \]

**Proof.** For \( X, Y \in D^\perp \) by using (4.21), we get
\[ g([X, Y], \xi) = g(\nabla_X Y, \xi) - g(\nabla_Y X, \xi) \]
\[ g([X, Y], \xi) = g(X, \nabla_Y \xi) - g(Y, \nabla_X \xi) = 2\beta \delta g(X, Y). \quad (5.2) \]

On the other hand, from (4.10), we have
\[ BP \nabla_X Y = -PA_{\phi^r} X - P\iota h(X, Y), \quad (5.3) \]

for \( X, Y \in D^\perp \). Then using lemma (4.5), we get from equation (5.3)
\[ BP [X, Y] = 0, \quad \text{for} \ X, Y \in D^\perp. \quad (5.4) \]
Theorem 5.2. Let $M$ be a generalised CR-submanifold of an $(\varepsilon, \delta)$-trans-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection. Then the distribution $D$ is never involutive.

Proof. For $X, Y \in D$ by using (4.19), we have

$$g([X, Y], \xi) = g(\nabla_X Y, \xi) - g(\nabla_Y X, \xi)$$

$$= 2\alpha\delta g(Y, BX) + 2\beta\delta g(X, Y) + g(X, PY) - g(Y, PX).$$

(5.5)

Taking $X \neq 0$ and $Y = BX$ in (5.5), it follows that $D$ is not involutive.

Next we have the following theorem.

Theorem 5.3. Let $M$ be a generalised CR-submanifold of an $(\varepsilon, \delta)$-trans-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection. Then the distribution $D \oplus \{\xi\}$ is involutive if and only if

$$h(BX, Y) - h(X, BY) + \nabla^+_Y CX - \nabla^+_X CY \in CD \oplus \nu$$

(5.6)

Proof. Applying $\phi$ to equation (4.14) and taking component in $D^+$, we have

$$Q\nabla_X Y = -Qt(h(X, BY) + \nabla^+_X CY - fh(X, Y))$$

for $X, Y \in D$.

Thus

$$Q[X, Y] = Qt(h(X, BY) - h(X, BY) + \nabla^+_Y CX - \nabla^+_X CY)$$

(5.7)

for $X, Y \in D$. Hence, the theorem follows from (5.7) and (4.34).

VI. Geometry of Leaves

Theorem 6.1. Let $M$ be a generalised CR-submanifold of an $(\varepsilon, \delta)$-trans-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection. Then the leaves of distribution $D^+$ are totally geodesic in $M$ if and only if

$$h(X, BZ) + \nabla^+_XCZ + \eta(CZ)X \in CD \oplus \nu$$

(6.1)

for $X \in D^+$ and $Z \in D \oplus \{\xi\}$.

Proof. For $X, Y \in D^+$ and $Z \in D \oplus \{\xi\}$ by using (2.2), (2.3), (3.8) and (4.8), we get

$$g(\nabla_X Z, Y) = -g(Y, \nabla_X Z) = -g(\nabla_X Z, Y) = -g(\phi\nabla_X Z, \phi X)$$

$$= g((\phi\nabla_X Z), \phi Y) - g(\nabla_X \phi Z, \phi Y) = -g(\nabla_X BZ + \nabla_X CZ, \phi Y)$$

$$= -g(h(X, BZ) + \nabla^+_X CZ + \eta(CZ)X, \phi Y).$$

(6.2)

Hence the theorem follows from (6.2).

Theorem 6.2. Let $M$ be a generalised CR-submanifold of an $(\varepsilon, \delta)$-trans-Sasakian manifold $\overline{M}$ with semi-symmetric non-metric connection. Then the distribution $D^+ \oplus \{\xi\}$ is involutive and its leaves are totally geodesic in $M$ if and only if

$$h(X, BY) + \nabla^+_XCY + \eta(CY)X \in CD \oplus \nu$$

(6.3)

for $X, Y \in D^+ \oplus \{\xi\}$.

Proof. For $X, Y \in D^+ \oplus \{\xi\}$ and $Z \in D^+$ by using (2.2), (2.3), (3.8), (4.7) and (4.8), we get

$$g(\nabla_X Z, Y) = g(\nabla_X Z, Y) = g(\phi\nabla_X Z, \phi Y) = g(\nabla_X \phi Z, \phi Z)$$

$$= g(\nabla_X BY + h(X, BY) - \eta(CY)X + \nabla^+_XCY, \phi Z).$$

(6.4)

Hence, the theorem follows from the equation (6.4).
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References