Common Fixed Point Theorem In Fuzzy Metric Space With Implicit Relation And Property (E.A.)

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Abstract: Aim of the present paper is to prove a common fixed point theorem for six maps via notion of pairwise commuting maps in fuzzy metric space satisfying contractive type implicit relation. Our result extends the result of Aalam, Kumar and Pant [1].

Keywords: Fuzzy Metric Space, weakly compatible maps, implicit relation, property (E.A.).

I. Introduction

Zadeh [21] introduced the concept of fuzzy sets in 1965 and in the next decade Kramosil and Michalek [10] introduced the concept of fuzzy metric spaces (briefly, FM-spaces) in 1975, which opened an avenue for further development of analysis in such spaces. Consequently in due course of time some metric fixed point results were generalized to FM-spaces by various authors viz George and Veeramani [5], Grabiec [6] and others.

For the last quarter of the twentieth century, there has been considerable interest to study the common fixed points of commuting maps and its weaker forms. In 1994, Mishra et al. [14] extended the notion of compatible maps (introduced by Jungck [8] in metric space) under the name of asymptotically commuting maps and Singh and Jain [19] extended the notion of weakly compatible maps (introduced by Jungck [9] in metric space) to FM-spaces. In 2007, Pant and Pant [16] extended the study of common fixed points of a pair of non-compatible maps (studied by Pant [15] in metric space) and the property (E.A) to FM-spaces. Note that the study of property (E.A) has been initiated by Aamri and Mouatwakil [2] as a generalization of the concept of non-compatible maps in metric spaces. Employing property (E.A), several results have been obtained in fuzzy metric space (see [1], [3], [11], [13]).

In 2009, Imdad et al. [7] introduced the notion of pairwise commuting maps. Implicit relations are used as a tool for finding common fixed point of contraction maps. Recently, Aalam, Kumar and Pant [1] proved a common fixed point theorem without completeness of space and continuity of involved mappings in FM-space, which generalizes the result of Singh and Jain [19].

In the present paper, we prove a common fixed point theorem for six self-maps in FM-space satisfying contractive type implicit relations. As an application, we extend our main result to four finite families of self-maps in FM-space.

II. Preliminaries

Definition 2.1 ([21]). Let X be any set. A fuzzy set A in X is a function with domain X and values in [0,1].

Definition 2.2 ([18]). A binary operation * : [0,1] × [0,1] → [0,1] is called a continuous t-norm if (0,1,* ) is an abelian topological monoid with the unit 1 such that a * b ≤ c * d , whenever a ≤ c and b ≤ d for all a, b, c, d ∈ [0,1].

Definition 2.3 ([10]). The triplet (X , M , *) is an FM-space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set in X²×(0:∞) satisfying the following conditions for all x, y, z ∈ X and t, s > 0,

1) M(x, y, t) = 1 for all t > 0 if and only if x = y;
2) M(x, y, 0) = 0;
3) M(x, y, t) = M(y, x, t);
4) M(x, y, t) * M(y, z, s) ≤ M(x, z, t + s);
5) M(x, y) : [0,∞) → [0,1] is left continuous.

Example 2.4. Let (X; d) be a metric space. Define a * b = ab(or a * b = min{a, b}) for all x, y ∈ X and...
Theorem 4. In our main result we extend this result and utilize the notion of commuting pairwise due to Imdad et al. [7].

IV. Main Result

Theorem 4.1 Let F,G,R,S, and T be self-maps of an FM-space (X, M, *) satisfying the following conditions:

(i) A(X) ⊂ T(X), B(X) ⊂ S(X).

(ii) (A, S) and (B, T) are weakly compatible pairs.

(iii) (A, S) or (B, T) satisfy the property (E.A.).

(iv) For some φ ∈ Φ and for all x, y ∈ X, t > 0.

φ(M(Ax, By, kt), M(Sx, Ty, t)) ≥ 0.

The range of one of the maps A, B, S and T is a complete subspace of X, than A, B, S and T have a unique common fixed point in X.

In our main result we extend this result and utilize the notion of commuting pairwise due to Imdad et al. [7].
(i) (F,SR) or (G, TH) satisfies the property (E.A);  
\[ \phi \left( M(Fx, Gy, t), M(SRx, THy, t) \right) \geq 0 \text{ for all } t > 0, x, y \in X \text{ and for some } \varphi \in \Phi . \]

(ii) \[ \phi \left( M(Fx, Gy, t), M(SRx, THy, t) \right) \geq 0 \text{ for all } t > 0, x, y \in X \text{ and for some } \varphi \in \Phi . \]

(iii) F(X) \subseteq TH(X) and G(X) \subseteq SR(X);

(iv) one of F(X), G(X), SR(X) AND TH(X) is a complete subspace of X.

Then the pair (F,SR) and (G,TH) have a unique common fixed point provided the pairs (F,SR) and (G,TH) commute pair wise i.e. FS = SF, GR = RG, SR = RS, GT = TG, GH = HG and TH = HT.

**Proof:** If the pair G, TH satisfies the property (E.A), then there exists a sequence \( \{ y_n \} \) in X such that \( G y_n \rightarrow z \) and \( THy_n \rightarrow z \), for some \( z \in X \) as \( n \rightarrow \infty \). Since G(X) \subseteq SR(X) there exists a sequence \( \{ x_n \} \) in X such that \( G y_n = SR x_n \). Hence, \( SR x_n \rightarrow z \) as \( n \rightarrow \infty \).

Now we show that \( F x_n \rightarrow z \) as \( n \rightarrow \infty \). By putting \( x = x_n \) and \( y = y_n \) in (ii), we have

\[ \phi \left( M(Fx_n, Gy_n, t), M(SRx_n, THy_n, t) \right) \geq 0. \]

Let \( F x_n \rightarrow l \) (\( \not\in X \)) for \( t > 0 \). Hence \( M(l, z, t) = 1 \). Thus \( Fz = \varphi(z) \) for all \( t > 0 \).

Using (\( \varphi \)), we get \( M(l, z, t) \geq 1 \) for all \( t > 0 \). Hence \( M(l, z, t) = 1 \).

Or

\[ \phi \left( M(Fu, z, t), M(z, z, t) \right) \geq 0. \]

On the other hand, since F(X) \subseteq TH(X) and \( Fz = z \), there exists a point \( v \in X \) such that THv = z. Now we show that THv = Gv. By putting \( x = u \) and \( y = v \) in (ii), we have

\[ \phi \left( M(Fu, z, t), M(SRu, THy, t) \right) \geq 0. \]

Letting \( n \rightarrow \infty \),

\[ \phi \left( M(Fu, z, t), M(SRu, THy, t) \right) \geq 0. \]

Using (\( \varphi \)), we get \( M(Fu, z, t) \geq 1 \) for all \( t > 0 \).

Thus \( Fu = z \). Hence \( Fu = SRu = z \) which shows that the pair (F, S) has a point of coincidence.

On the other hand, since F(X) \subseteq TH(X) and \( Fu = z \), there exists a point \( v \in X \) such that THv = z.

Now we show that \( THv = Gv \). By putting \( x = u \) and \( y = v \) in (ii), we have

\[ \phi \left( M(Fu, z, t) \right) \geq 0. \]

Using (\( \varphi \)), we get \( M(Fu, z, t) \geq 1 \) for all \( t > 0 \).

Thus \( Fu = SRu = z \) which shows that the pair (G, T) has a point of coincidence.

Since the pairs (F, SR) and (G, TH) are commuting pair wise i.e. FS = SF, FR = RF, SR = RS, GT = TG, GH = HG and TH = HT. It implies that both the pairs (F, SR) and (G, TH) are weakly compatible at \( u \) and \( v \) respectively.

Therefore, \( Fu = SRu = Gv = THv \). Hence \( Fz = F(SR)u = (SR)Fu = (SR)z \) and \( Gz = G(TH)v = (TH)Gv = THz \). Now we assert that \( z \) is a fixed point of the maps F, S and R. Putting \( x = z \) and \( y = z \) in (ii), we have

\[ \phi \left( M(Fz, G(z), t) \right) \geq 0. \]

And so

\[ \phi(M(z, Hz, t), M(z, z, t)) \geq 0 \text{ or } \phi(M(z, Hz, t), M(z, z, t)) \geq 0. \]
Using \(\phi_2\) we get \(M(z, Hz, t) \geq 1\) for all \(t > 0\). Hence \(M(z, Hz, t) = 1\). Thus conclude that \(z\) is a common fixed point of self-maps \(F, G, R, S, H, T\). Let \(w\) be another common fixed point of self-maps \(F, G, R, S, H, T\) and then on using (ii) with \(x = z, y = w\), we have

\[
\phi(M(Fz, Bw, t), M(Sz, Tw, t), M(FzSz, t), M(Gw, Tw, t)) \geq 0
\]

or

\[
\phi(M(z, w, t), M(z, w, t), M(z, z, t), M(w, w, t)) \geq 0
\]

Using \(\phi_2\), we get \(M(z, w, t) \geq 1\) for all \(t > 0\). Hence, \(M(z, w, t) = 1\). Therefore, \(z = w\) and the common fixed point is unique.

We can also prove the same result if the pair \((F, S)\) satisfies the property \((E.A)\). The proof is similar when \(TH(X)\) is assumed to be a complete subspace of \(X\). The remaining two cases pertain essentially to the previous cases. If we assume that \(F(X)\) is a complete subspace of \(X\), then \(z \in F(X) \subset TH(X)\) or \(G(X)\) is a complete subspace of \(X\), then \(z \in G(X) \subset SR(X)\).

Thus we can establish that both the pairs \((F, S)\) and \((G, TH)\) have a point of coincidence each. This completes our proof.

On taking \(R = H = IX\) (the identity maps on \(X\)) in Theorem 4.2 we get the result of Aalam et al. [1].

**Corollary 4.2** ([1] Theorem 4.1) Let \(A, B, S\) and \(T\) be self maps of an FM-Space \((X, M,*\)) satisfying

(i) \((A, S)\) or \((B, T)\) satisfies the property \((E.A)\)

(ii) \(\phi(M(Ax, By, t), M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t)) \geq 0\), for all \(t > 0\) and \(x, y \in X\) and some \(\phi \in \Phi\).

(iii) \(A(X) \subseteq T(X), B(X) \subseteq S(X)\)

(iv) One of \((A, X), (B, X), (S, X)\) and \((T, X)\) is a complete subspace of \(X\).

Then the pairs \((A, S)\) and \((B, T)\) have a point of coincidence each. Moreover, \(A, B, S, T\) have a unique common fixed point provided both the pairs \((A, S)\) and \((B, T)\) are weakly compatible.

Now we give an example which illustrates Corollary 4.2.

**Example 4.3** Let \(X = \{2, 20\}\) and \(d\) be the usual metric on \(X\). For each \(t \in [0, \infty)\) define

\[
M(x, y, t) = \begin{cases} 
\frac{t}{t + |x - y|} & \text{if } t > 0 \\
0 & \text{if } t = 0
\end{cases}
\]

For all \(x, y \in X\), clearly \((X, M, *)\) is an FM-Space, where \(*\) is defined as \(a^*b = ab\).

Let \(\Phi : (R^*)^4 \to R\), be defined as in example 3.1 and define the self maps \(A, B, S\) and \(T\) by

\[
A(x) = \begin{cases} 
2, & \text{if } x \geq 2, \\
6, & \text{if } x < 2
\end{cases}
\]

\[
B(x) = \begin{cases} 
2, & \text{if } x = 2 \text{ or } x > 5, \\
6, & \text{if } 2 < x \leq 5
\end{cases}
\]

\[
T(x) = \begin{cases} 
2, & \text{if } x = 2, \\
12, & \text{if } 2 < x \leq 5, \\
(x + 1)/3, & \text{if } x > 5
\end{cases}
\]

Then \(A, B, S, T\) satisfy all the conditions of Corollary 4.2 and have a unique common fixed point \(x = 2\). Clearly, the pairs \((A, S)\) and \((B, T)\) are noncompatible. We assume that \(\{x_n\}\) is a sequence defined as \(x_n = 5 + 1/n, \ n \geq 1\). Also, the pairs \((A, S)\) and \((B, T)\) are weakly compatible since they commute at their coincidence points. It can also be seen that \(B\) and \(T\) satisfy the property \((E.A)\) and all the maps \(A, B, S, T\) are discontinuous at the common fixed point.

On taking \(A = B\) and \(S = T\) in Corollary 4.2, we get the following result:

**Corollary 4.4** Let \(A\) and \(S\) be self-maps of an FM-space \((X, M,*\)) satisfying

1. \((A, S)\) satisfies the property \((E.A)\);
2. For all \(t > 0\); \(x, y \in X\) and for some \(\phi \in \Phi\);
3. \(A(X) \subseteq S(X)\);
4. One of \((A, X)\) and \((S, X)\) is a complete subspace of \(X\).

Then the pair \((A, S)\) has a point of coincidence. Moreover, \(A\) and \(S\) have a unique common fixed point provided the pair \((A, S)\) is weakly compatible.

As an application of Corollary 4.2, we extend the related result to four families of self-maps on FM-spaces.
Theorem 4.5. Let \( \{ A_1, A_2, \ldots, A_n \}, \{ B_1, B_2, \ldots, B_q \}, \{ S_1, S_2, \ldots, S_n \}\) and \( \{ T_1, T_2, \ldots, T_q \} \) be four finite families of self-maps of an FM-space \((X, M, *)\) such that \( A_i = A_1, A_2, \ldots, A_n \), \( B_i = B_1, B_2, \ldots, B_q \), \( S_i = S_1, S_2, \ldots, S_n \) and \( T_i = T_1, T_2, \ldots, T_q \) which satisfy conditions (i)-(iv) of corollary 4.2. Then the pairs \((A, S)\) and \((B, T)\) have a point of coincidence each. Moreover, if the family \( \{ A_i \} \) \((i = 1,2,\ldots,n)\) commute pairwise with the family \( \{ S_i \} \) \((j = 1,2,\ldots,n)\), whereas the family \( \{ B_i \} \) \((r = 1,2,\ldots,p)\) commute pairwise with the family \( \{ T_i \} \) \((k = 1,2,\ldots,q)\) then for all \( \{ i \in \{ 1,2,\ldots,m \} \}, \{ j \in \{ 1,2,\ldots,n \} \}, \{ r = 1,2,\ldots,p \}, \{ k = 1,2,\ldots,q \} \) \( A_i \cup B_r \cup S_j \cup T_k \) have a common fixed point.

Proof. Using the terminology of Theorem 4.1, the proof of this theorem is similar to that of Theorem 3.1 contained in [7], hence it is omitted.

Remark 4.6. Theorem 4.5 improves and extends the results of Singh and Jain [19] and Aalam et al. [1] to four finite families of self-maps. 

By setting \( A_1 = A_2 = \ldots = A_n = A, B_1 = B_2 = \ldots = B_q = B, S_1 = S_2 = \ldots = S_n = S \) and \( T_1 = T_2 = \ldots = T_q = T \) in Theorem 4.5, we deduce the following:

Corollary 4.7. Let \( A, B, S \) and \( T \) be self-maps of an FM-space \((X, M, *)\) satisfying: 

1. \((A^n, S^n) \) or \((B^n, T^n)\) satisfies the property \((E.A)\); 
2. \( M \left( A^* x, B^* y, t \right) \), \( M \left( S^* x, T^* y, t \right) \) satisfies the property \((E.A)\); 
3. \((A^m) \subseteq T^n(X), (B^m) \subseteq S^n(X)\); 
4. One of \((A^m), (B^m), (S^n)\) and \((T^n)\) is a complete subspace of \(X\). 

Then the pairs \((A^m, S^n)\) and \((B^m, T^n)\) have a point of coincidence each. Moreover \( A, B, S \) and \( T \) have a unique common fixed point provided both the pairs \((A^m, S^n)\) and \((B^m, T^n)\) commute.

Remark 4.8. From the results, it is asserted that property \((E.A)\) buys containment of ranges without any continuity requirements, besides minimize the commutativity conditions of the maps to the commutativity at the points of coincidence. More- over, property \((E.A)\) allows replacing the completeness requirement of completeness of the range space.

V. Conclusion

We established a common fixed point theorem in fuzzy metric space along with property \((E.A)\). and implicit relation. In this theorem we have used the mappings which are pairwise commutative. The result has a number of applications in various branch of Mathematics and Mathematical Sciences.

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