Multidimensional integral transform involving I-function of several complex variables as kernel

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**Abstract:** The theory of integral transforms is very useful in solving various types of boundary value problems. By giving various values to the kernel \(k(x; s)\) and considering the interval \((0, 1)\) generally, a number of integral transforms have been introduced and studied by several authors from time to time. In the present paper we have introduced multidimensional I-function transform involving \(I\)-function of \(r\) variables as kernel. In our earlier paper we have established the multiple Mellin transform of product of two \(I\)-functions. With the help of that result, in the present paper we have discussed some theorems on multidimensional \(I\)-function transform. Special cases include the results involving \(H\)-function of several complex variables, \(I\)-function of two variables and \(H\)-function of two variables. The special case also includes the results given by V.C.Nair. 

**Keywords:** Mellin-Barnes integral, \(H\)-function, \(H\)-function transform, \(I\)-function, Multidimensional \(I\)-function transform

I. Introduction

In 1997, Rathie introduced the generalization of the well-known Fox’s \(H\)-function [1] which has very recently found interesting applications in wireless communication ([2],[3],[4]). Motivated by the \(I\)-function, Shantha Kumari, Nambisan and Rathie introduced \(I\)-function of two variables [5] which is a natural generalization of the \(H\)-function of two variables introduced earlier by Mittal and Gupta [6] and discussed some of its important properties. Very recently we have introduced and studied an extension of the \(I\)-function of several complex variables. This function is defined and represented in the following manner [7].

\[
I[z_1,...,z_r] = \int_{L_1}^{L_r} ... \int_{L_1}^{L_r} \phi(s_1,...,s_r) \theta(s_i) z_1^{s_1}...z_r^{s_r} ds_1...ds_r
\]

(1)

where \(\phi(s_1,...,s_r)\) and \(\theta(s_i), i=1,...,r\) are given by

\[
\phi(s_1,...,s_r) = \prod_{j=1}^{n} \Gamma_{A_j} \left( 1-a_j + \sum_{i=1}^{r} a_j(i)s_i \right)
\]

(2)

\[
\theta(s_i) = \prod_{j=1}^{m} \Gamma_{D_j} \left( c_j(i) - \sum_{i=1}^{r} \gamma_j(i)s_i \right)
\]

(3)

Also

- \(z_i \neq 0\) for \(i = 1,...,r\)
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- an empty product is interpreted as unity.
- the parameters \( m_j, n_j, p_j, q_j \) \( (j=1,...,r) \), \( n, p, q \) are nonnegative integers such that \( 0 \leq n \leq p, q \geq 0, 0 \leq n_j \leq p_j, 0 \leq m_j \leq q_j \) \( (j=1,...,r) \) (not all zero simultaneously).
- \( \alpha_j^{(i)} \) \( (j=1,...,p; i=1,...,r) \), \( \beta_j^{(i)} \) \( (j=1,...,q; i=1,...,r) \), \( \gamma_j^{(i)} \) \( (j=1,...,p_i; i=1,...,r) \), \( \delta_j^{(i)} \) \( (j=1,...,q_i; i=1,...,r) \) are assumed to be positive quantities for standardization purpose.
- The parameters \( j_{m,n,p,q} \) \( j=1,...,r \), \( n, p, q \) are nonnegative integers such that \( 0 \leq n \leq p, q \geq 0, 0 \leq m \leq q \) \( j=1,...,r \) (not all zero simultaneously).
- \( \alpha_j \) \( (j=1,...,p) \), \( b_j \) \( (j=1,...,q) \), \( c_j^{(i)} \) \( (j=1,...,p_i; i=1,...,r) \), \( d_j^{(i)} \) \( (j=1,...,q_i; i=1,...,r) \) are complex numbers.
- The parameters \( \alpha_j \) \( (j=1,...,p) \), \( B_j \) \( (j=1,...,q) \), \( C_j^{(i)} \) \( (j=1,...,p_i; i=1,...,r) \), \( D_j^{(i)} \) \( (j=1,...,q_i; i=1,...,r) \) of various gamma functions involved in (2) and (3) may take non-integer values.
- The contour \( L_i \) in the complex \( s \)-plane is of Mellin–Barnes type which runs from \( c_i \rightarrow \infty \) to \( c_i+\infty \) with indentations, if necessary, in such a manner that all singularities of \( \Gamma^{(i)} \) \( (j=1,...,m_i; i=1,...,r) \) and \( \Gamma^{(i)} \) \( (j=1,...,m_i; i=1,...,r) \) are to the right and \( \Gamma^{(i)} \) \( (j=1,...,n_i; i=1,...,r) \) are to the left of \( L_i \), \( i=1,\ldots,r \).

Following the results of Braaksma[8] the I-function of `r' variables is analytic if

\[ \mu_i = \sum_{j=1}^{p} \alpha_j^{(i)} + \sum_{j=1}^{q} B_j^{(i)} - \sum_{j=1}^{p} C_j^{(i)} - \sum_{j=1}^{q} D_j^{(i)} \leq 0, \quad i=1,...,r \]  

(4)

1.1 Convergence Conditions

The integral (1) converges absolutely if

\[ \arg(z_k) \leq \frac{1}{2} \Delta_k - \kappa, \quad k=1,...,r \]  

(5)

where \( \Delta_k = \begin{pmatrix} \frac{p}{2} \sum j=1 \alpha_j^{(k)} + \sum j=1 \beta_j^{(k)} + \sum j=1 \delta_j^{(k)} - \sum j=1 \gamma_j^{(k)} \end{pmatrix} + \begin{pmatrix} \frac{q}{2} \sum j=1 \alpha_j^{(k)} - \sum j=1 \beta_j^{(k)} + \sum j=1 \delta_j^{(k)} - \sum j=1 \gamma_j^{(k)} \end{pmatrix} > 0 \]  

(6)

And if

\[ \arg(z_k) \leq \frac{1}{2} \Delta_k - \kappa, \quad \Delta_k > 0, \quad k=1,...,r \]  

then integral (1) converges absolutely under the following conditions.

(i) \( \mu_k = 0, \quad \Omega_k \leq -1 \) where \( \mu_k \) is given by (4) and

\[ \Omega_k = \sum_{j=1}^{p} \left[ \frac{1}{2} - R \left( a_j \right) \right] + \sum_{j=1}^{q} \left[ \frac{1}{2} - R \left( b_j \right) \right] - \sum_{j=1}^{p} \left[ \frac{1}{2} - R \left( c_j^{(k)} \right) \right] - \sum_{j=1}^{q} \left[ \frac{1}{2} - R \left( d_j^{(k)} \right) \right] > 0 \]  

(7)

(ii) \( \mu_k \neq 0 \) with \( \sigma_k = \tau_k \) \( (k=1,...,r) \). \( \sigma_k \) and \( \tau_k \) are so chosen that for \( |k| \rightarrow \infty \) we have \( \Omega_k + \sigma_k \mu_k < -1 \).

We have discussed the proof of convergent conditions in our earlier paper [7].

Remark 1

If \( D_j^{(i)} = 1 \) \( (j=1,...,m_i; i=1,...,r) \) in (1), then the function will be denoted by
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\[ \tau\left[z_1, \ldots, z_r\right] = \int_{p_1 \leq p_2 \leq \ldots \leq \pi_1} \int_{\Gamma_{1-d} + \gamma_s \Gamma_{1-c} + \gamma_s} \left[ z_1 \left( a_j \alpha^{(i)} \beta_j \gamma^{(r)} \right) \right] \tau_{\Gamma_{1-d} + \gamma_s \Gamma_{1-c} + \gamma_s} \left[ \phi_{\Gamma_{1-d} + \gamma_s \Gamma_{1-c} + \gamma_s} \right] \left| z_1 \right| \left( a_j \alpha^{(i)} \beta_j \gamma^{(r)} \right) \left| \phi_{\Gamma_{1-d} + \gamma_s \Gamma_{1-c} + \gamma_s} \right| dz_1 \ldots dz_r \]

where

\[ \left| \phi_{\Gamma_{1-d} + \gamma_s \Gamma_{1-c} + \gamma_s} \right| = \prod_{j=1}^{r} \Gamma_j \left( \frac{\alpha}{\beta} \right) u_j \Gamma_j \left( \frac{\alpha}{\beta} \right) u_j + 1 \]

Remark 2

If \( C_j^{(i)} = 1 (j = 1, \ldots, n_r, i = 1, \ldots, r) \) and \( n_0 = 0 \) in (1), then the function will be denoted by

\[ T_j\left[z_1, \ldots, z_r\right] = \int_{p_1 \leq p_2 \leq \ldots \leq \pi_1} \int_{\Gamma_{1-d} + \gamma_s \Gamma_{1-c} + \gamma_s} \left[ z_1 \left( a_j \alpha^{(i)} \beta_j \gamma^{(r)} \right) \right] \tau_{\Gamma_{1-d} + \gamma_s \Gamma_{1-c} + \gamma_s} \left[ \phi_{\Gamma_{1-d} + \gamma_s \Gamma_{1-c} + \gamma_s} \right] \left| z_1 \right| \left( a_j \alpha^{(i)} \beta_j \gamma^{(r)} \right) \left| \phi_{\Gamma_{1-d} + \gamma_s \Gamma_{1-c} + \gamma_s} \right| dz_1 \ldots dz_r \]

where

\[ \phi_{\Gamma_{1-d} + \gamma_s \Gamma_{1-c} + \gamma_s} = \prod_{j=1}^{r} \Gamma_j \left( \frac{\alpha}{\beta} \right) u_j \Gamma_j \left( \frac{\alpha}{\beta} \right) u_j + 1 \]

Definition

Mellin transform of a function \( f(x) \) is defined as

\[ M\left[f\left(x\right)\right] = \int_0^\infty \phi_j x^{\alpha-1} f\left(x\right) dx \]
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II. Result Used

The following important theorem which is an analogue of the well-known Parseval-Goldstein theorem will be required in the sequel [9]:

**Theorem**

If \( T\{f_1(x);s\} = \varphi_1(s) \) and \( T\{f_2(x);s\} = \varphi_2(s) \)

then

\[
\int_0^\infty f_1(x) \varphi_2(x) \, dx = \int_0^\infty f_2(x) \varphi_1(x) \, dx
\]

(14) provided

that the various integrals involved converge absolutely.
Gupta and Mittal\[10\] have proved the following theorem.

**Uniqueness theorem**

If \( f_1(x) \) and \( f_2(x) \) are continuous for \( x \geq 0 \) and

\[
\int_0^\infty H_{p,q}^{m,n} \left[ \begin{array}{c} a_j; a_l \; (l) \; (r) \\ b_j; b_l \; (l) \; (r) \end{array} \right] f_1(x) \, dx = \int_0^\infty H_{p,q}^{m,n} \left[ \begin{array}{c} a_j; a_l \; (l) \; (r) \\ b_j; b_l \; (l) \; (r) \end{array} \right] f_2(x) \, dx
\]

both integrals being convergent, then \( f_1(x) \equiv f_2(x) \) \( (15) \)

Without loss of generality the above theorem can be extended to multiple integrals with appropriate convergent conditions.

We have proved the following result \[11\].

\[
\begin{align*}
\int_0^\infty \int_0^\infty \cdots \int_0^\infty & \left[ s_1 x_1^{\lambda_1}, \ldots, s_x x_r^{\lambda_r} \right] \times \left[ t_1 x_1^{\mu_1}, \ldots, t_r x_r^{\mu_r} \right] \, dx_1 \cdots dx_r \\
& = \frac{1}{\mu_1 \cdots \mu_r} \left[ \frac{\lambda_1}{\mu_1}, \ldots, \frac{\lambda_r}{\mu_r} \right] \left[ C; C_1; \ldots; C_r \\ D; D_1; \ldots; D_r \right]
\end{align*}
\]

(17)

where

\[
\begin{align*}
C &= \left( a_j; a_l \; (l) \; (r) ; A_j \right) \\
C_1 &= \left( c_{j,i}^{(l)} ; \gamma_{j,i}^{(l)} ; 1 \right) \\
D &= \left( b_j; b_l \; (l) \; (r) ; B_j \right)
\end{align*}
\]

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\[ D_i = \left( \frac{d(i) \delta(i); 1}{m_i} \right) \left( 1 - c(i) \gamma(i) \frac{\rho_i}{\mu_i} \delta(i) \frac{\lambda_i}{\rho_i} ; 1 \right) \left( \frac{D_j(i)}{n_j+1} \right), \quad i = 1, \ldots, r \]

provided that

(i) $\lambda_i, \mu_i > 0, i = 1, 2, \ldots, r$

(ii) $-\lambda_i \min_{1 \leq j \leq m_i} \Re \left( \frac{d(j)}{\delta(j)} \right) - \mu_i \min_{1 \leq j \leq n_i} \Re \left( \frac{d(j)}{\gamma(j)} \right) < \Re(\rho_i)$

and

(iii) the $I$-functions involved in (17) exist.

Definition

In an attempt to generalize the $H$-function transform we introduced an integral transform whose kernel is the $I$-function of several complex variables defined by (1). This integral transform is defined and represented in the following manner:

\[ \varphi(p_1, \ldots, p_r) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \left[ c_1(p_1 t_1)^{\gamma_1} \cdots c_r(p_r t_r)^{\gamma_r} \right] f(t_1, \ldots, t_r) dt_1 \cdots dt_r \]

provided that the integral on the right hand side of (18) is absolutely convergent.

III. Some Theorems On Multidimensional I-Function Transform

Theorem 1

If

\[ \varphi(p_1, \ldots, p_r) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \left[ c_1(p_1 t_1)^{\gamma_1} \cdots c_r(p_r t_r)^{\gamma_r} \right] f(t_1, \ldots, t_r) dt_1 \cdots dt_r \]

and

\[ \psi(p_1, \ldots, p_r) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \left[ c_1(p_1 t_1)^{\gamma_1} \cdots c_r(p_r t_r)^{\gamma_r} \right] f(t_1^{\gamma_1}, \ldots, t_r^{\gamma_r}) h(t_1, \ldots, t_r) dt_1 \cdots dt_r \]

then

\[ \psi(p_1, \ldots, p_r) = \int_0^\infty \int_0^\infty \cdots \int_0^\infty g(x_1, \ldots, x_r, p_1, \ldots, p_r) \varphi(x_1, \ldots, x_r) dx_1 \cdots dx_r \]

provided the integrals involved are absolutely convergent.
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Proof

Using (14) for the transform pairs (19) and (22),

\[
\int_0^\infty \cdots \int_0^\infty g(t_1,...,t_r,k_1,...,k_r) \phi(t_1,...,t_r) dt_1...dt_r = \int_0^\infty \cdots \int_0^\infty t_1^{r-1}...t_r^{r-1} h(t_1,...,t_r)
\]

\[
\mathcal{I}^r \left[ c_1(t_1^{r_1}k_1),...,c_r(t_r^{r_r}k_r) \right] f(t_1,...,t_r) dt_1...dt_r
\]

\[
\int_0^\infty \cdots \int_0^\infty g(x_1,...,x_r,p_1,...,p_r) \phi(x_1,...,x_r) dx_1...dx_r = \int_0^\infty \cdots \int_0^\infty t_1^{r_1}...t_r^{r_r} h(t_1,...,t_r)
\]

\[
\mathcal{I}^r \left[ c_1(t_1^{r_1}k_1),...,c_r(t_r^{r_r}k_r) \right] f(t_1,...,t_r) dt_1...dt_r
\]

Hence,

Putting \( t_i^r = x_i, i=1,...,r \) on the right hand side and using (20) the result follows.

Theorem 2

If \( \phi(p_i^{r_i},...,p_r^{r_r}) = \int_0^\infty \cdots \int_0^\infty c_1(p_1t_1)^{r_1},...,c_r(p_rt_r)^{r_r} f(t_1,...,t_r) dt_1...dt_r \) \hspace{1cm} (23)

and \( \psi(p_1,...,p_r) = \int_0^\infty \cdots \int_0^\infty x_1^{r_1-1}...x_r^{r_r-1} I_c(p_1x_1)^{r_1},...,I_c(p_rx_r)^{r_r} \phi(x_1,...,x_r) dx_1...dx_r \) \hspace{1cm} (24)

then

\[
\psi(p_1,...,p_r) = \frac{1}{(v_1...v_r)(\lambda_1...\lambda_r)} c_1^{r_1/v_1}...c_r^{r_r/v_r} \int_0^\infty \cdots \int_0^\infty t_1^{r_1/v_1}...t_r^{r_r/v_r}
\]

\[
C_l p_l \left( c_1^{1/v_1},...,c_r^{1/v_r} \right) \sigma_1^{r_1/v_1}...\sigma_r^{r_r/v_r} f(t_1,...,t_r)
\]

\[
E_i = \left( a_j;\alpha_j^{(i)},...,\alpha_j^{(r)};A_j^i \right)_p, \left( 1-b_j - \sum_{i=1}^r b_j^{(i)} \frac{\rho_j}{v_j^{(i)}},\beta_j^{(i)};\sigma_j^{(i)},\beta_j^{(i)};\sigma_j^{(i)};B_j \right)_q
\]

\[
E_i = \left( c_j^{(i)};\gamma_j^{(i)},\rho_j^{(i)};1 \right)_m, \left( 1-d_j^{(i)}-\delta_j^{(i)};\frac{\rho_j}{v_j^{(i)}},\sigma_j^{(i)};1 \right)_m
\]

\[
F_i = \left( b_j^{(i)};\beta_j^{(i)},...,\beta_j^{(r)};B_j^i \right)_q, \left( 1-a_j - \sum_{i=1}^r a_j^{(i)} \frac{\rho_j}{v_j^{(i)}},\alpha_j^{(i)};\sigma_j^{(i)},\alpha_j^{(i)};\sigma_j^{(i)};A_j \right)_p
\]
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\[
\mathbf{F}_i = \left( d_j^{(i)}, \delta_j^{(i)}; 1 \right) m_i, \left( 1 - c_j^{(i)} \gamma_j^{(i)} p_i v_j^{-1} \gamma_j^{(i)} \sigma_i v_j^{-1}; 1 \right) n_i, \\
\left( 1 - c_j^{(i)} \gamma_j^{(i)} p_i v_j^{-1} \gamma_j^{(i)} \sigma_i v_j^{-1}; C_j^{(i)} \right) p_i, m_{i+1} \left( d_j^{(i)} \delta_j^{(i)}; D_j^{(i)} \right) q_i \quad i = 1, ..., r
\]

provided 
\[\sigma > 0, \lambda > 0, \Delta > 0, \Delta' > 0\]

where 
\[
\Delta = \beta - \alpha; \Delta' = \beta - \alpha
\]

Proof

On the R.H.S of (24) use (23) to get

\[
\psi(p_1, ..., p_r) = \int_0^\infty \int_0^\infty \int_0^\infty \ldots \int_0^\infty x_1^{p_1-1} x_r^{p_r-1} \mathbf{T} \left[ C_1(p_1 x_1)^{\alpha_1}, \ldots, C_r(p_r x_r)^{\alpha_r} \right] \\
\times \mathbf{T} \left[ c_1(x_1^r t_1^\lambda \ldots c_r(x_r^r t_r^\lambda) \right] dx_1 \ldots dx_r f(t_1, ..., t_r) dt_1 \ldots dt_r
\]

(26)

In (26) use (17) to get the R.H.S of (25).

Theorem 3

If 
\[
\varphi(p_1, ..., p_r) = \int_0^\infty \int_0^\infty \int_0^\infty \ldots \int_0^\infty k(x_1, ..., x_r, t_1, ..., t_r) f(x_1, ..., x_r) dx_1 \ldots dx_r
\]

(27)

and

\[
\psi(p_1, ..., p_r) \varphi(p_1, ..., p_r) = \int_0^\infty \int_0^\infty \int_0^\infty \ldots \int_0^\infty C_1(p_1 t_1)^{\mu_1}, \ldots, C_r(p_r t_r)^{\mu_r} \mathbf{T} \left[ g(t_1, ..., t_r) dt_1 \ldots dt_r \right]
\]

(28)

then

\[
g(t_1, ..., t_r) = \int_0^\infty \int_0^\infty \ldots \int_0^\infty \right. \mathbf{T} \left[ C_1(p_1 t_1)^{\mu_1}, \ldots, C_r(p_r t_r)^{\mu_r} \right] \mathbf{K} \left[ \ldots \right] \mathbf{T} \left[ C_1(p_1 t_1)^{\mu_1}, \ldots, C_r(p_r t_r)^{\mu_r} \right] \mathbf{K} \left[ \ldots \right] \mathbf{T} \left[ C_1(p_1 t_1)^{\mu_1}, \ldots, C_r(p_r t_r)^{\mu_r} \right] \mathbf{K} \left[ \ldots \right] \mathbf{T} \left[ C_1(p_1 t_1)^{\mu_1}, \ldots, C_r(p_r t_r)^{\mu_r} \right] \mathbf{K} \left[ \ldots \right] \mathbf{T} \left[ C_1(p_1 t_1)^{\mu_1}, \ldots, C_r(p_r t_r)^{\mu_r} \right] \mathbf{K} \left[ \ldots \right]
\]

(29)

provided the integrals involved are absolutely convergent.

Proof

Multiplying both sides of (30) by \( f(x_1, ..., x_r) \) and integrating with respect to \( x_1, ..., x_r \) from 0 to \( \infty \),

\[
\psi(p_1, ..., p_r) \int_0^\infty \int_0^\infty \int_0^\infty \ldots \int_0^\infty C_1(p_1 x_1)^{\mu_1}, \ldots, C_r(p_r x_r)^{\mu_r} f(x_1, ..., x_r) dx_1 \ldots dx_r
\]

\[
= \int_0^\infty \int_0^\infty \ldots \int_0^\infty \mathbf{T} \left[ C_1(p_1 t_1)^{\mu_1}, \ldots, C_r(p_r t_r)^{\mu_r} \right] dt_1 \ldots dt_r dx_1 \ldots dx_r
\]

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Using (27), L.H.S of (31) is \( \psi(p_1,...,p_r) \phi(p_1',...,p_r') \). Therefore from (17) and (28) we get (29).

The change in the order of integration is valid provided the \( I \)-function transform of \( \left[ g(t_1,...,t_r) \right] \) and \( \left| k(x_1,...,x_r; t_1,...,t_r) \right| \) exists and the integral (29) is convergent by virtue of De la Vallee Pousein’s theorem [12, p-504].

**Special Cases**

In above three theorems, if we take \( A_{ij} = B_{ij} = C_{ij} = D_{ij} = 1 \), we get the corresponding results involving \( H \)-function of \( r \) variables. If we put \( r = 2 \) in above three theorems, they reduce to the corresponding results involving \( I \)-function of two variables. Further putting all exponents unity we get the results involving \( H \)-function of two variables.

When \( p = q = 0 \) and \( r = 1 \) in above three theorems we obtain the results given by Nair [13].

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