On generalized Weyl’s type theorem

A. Babbah¹, M. Zohry²

¹²(University Abdelmalek Essaadi, Faculty of Sciences, Mathematics Department, BP. 2121, Tetouan, Morocco)

Abstract: It is shown that if a bounded linear operator T or its adjoint T* has the single-valued extension property, then generalized Browder’s theorem holds for f(T) for every f ∈ H(σ(T)). We establish the spectral theorem for the B-Weyl spectrum which generalizes [15, Theorem 2.1] and we give necessary and sufficient conditions for such operator T to obey generalized Weyl’s theorem.

Keywords: Single-valued extension property, Fredholm theory, generalized Weyl’s theorem, generalized Browder’s theorem.

INTRODUCTION AND NOTATIONS

Let X denote an infinite-dimensional complex Banach space and L(X) the unital (with unit the identity operator, I, on X) Banach algebra of bounded linear operators acting on X. For an operator T ∈ L(X) write T* for its adjoint, N(T) for its null space, R(T) for its range, σ(T) for its spectrum, σ_n(T) for its approximate point spectrum, α(T) for its nullity and β(T) for its defect.

T is called an upper semi-Fredholm (resp. a lower semi-Fredholm) operator if the range R(T) of T is closed and α(T) < ∞ (resp. β(T) < ∞). A semi-Fredholm operator is an upper or a lower semi-Fredholm operator. If both α(T) and β(T) are finite, then T is called a Fredholm operator and the index of T is defined by ind(T) = α(T) − β(T).

For a T -invariant closed linear subspace Y of X, let T / Y denote the operator given by the restriction of T to Y.

For a bounded linear operator T and for each integer n, define T_n to be the restriction of T to R(T^n) viewed as a map from R(T^n) into itself. If for some integer n the range R(T^n) is closed and T_n = T / R(T^n) is a Fredholm (resp. semi-Fredholm) operator, then T is called a B-Fredholm (resp. semi-B-Fredholm) operator. In this case, from [3, Proposition 2.1] T_m is a Fredholm operator and ind(T_m) = ind(T_n) for each m ≥ n. This permits to define the index of a B-Fredholm operator T as the index of the Fredholm operator T_n where, n is any integer such that R(T^n) is closed and T_n is a Fredholm operator. It is shown (see [2, Theorem 3.2]) that if S and T are two commuting B-Fredholm operators then the product ST is a B-Fredholm operator and BF(X) be the class of all B-Fredholm operators and BF(T) be the B-Fredholm resolvent of T and let σ_BF(T) be the B-Fredholm spectrum of T. The class BF(X) has been studied by M. Berkani (see [3, Theorem 2.7]) where it was shown that an operator T ∈ L(X) is a B-Fredholm operator if and only if T = S_0 ⊕ S_1 where S_0 is a Fredholm operator and S_1 is a nilpotent one. He also proved that σ_BF(T) is a closed subset of σ(T) contained in the spectrum σ(T) and showed that the spectral mapping theorem holds for σ_BF(T), that is, for any complex-valued analytic function on a neighborhood of σ(T) (see [3, Theorem 3.4]). From [21] we recall that for T ∈ L(X), the ascent a(T) and the descent d(T) are given by

And
respectively, where the infimum over the emptyset is taken to be \(\infty\). If \(a(T)\) and \(d(T)\) are both finite the \(a(T) = d(T) = \infty\). The set of all complex \(T\) for every \(n \in \mathbb{N}\). The semi-regular resolvent set is defined by \(s\). We note that \(s\) is an open subset of \(\mathbb{C}\). The semi-B-Fredholm resolvent set of \(T\) is given by \(a(T) = d(T) = \infty\).

We recall that an operator \(T \in \mathcal{L}(X)\) has the single-valued extension property, abbreviated SVEP, if, for every open set \(U \subseteq \mathbb{C}\), the only analytic solution \(f: U \rightarrow X\) of the equation \((T - \lambda)(f) = 0\) for all \(\lambda \in U\) is the zero function on \(U\). We will denote by \(\mathcal{H}(\sigma(T))\) the set of all complex-valued functions which are analytic on an open set containing \(\sigma(T)\). As a consequence of [9, Théorème2.7], we obtain the following result.

**Proposition 1** Let \(T \in \mathcal{L}(X)\).

(i) If \(T\) has the SVEP then \(\sigma(T) = \sigma(T)\).

(ii) If \(T^*\) has the SVEP then \(\sigma(T) = \sigma(T)\).

For our investigations we need the following result.

**Proposition 2** Let \(T \in \mathcal{L}(X)\).

(i) If \(T\) has the SVEP then \(\text{ind}(T) \leq 0\) for every \(\lambda \in \rho_{\mathbb{C}}(T)\).

(ii) If \(T^*\) has the SVEP then \(\text{ind}(T) \geq 0\) for every \(\lambda \in \rho_{\mathbb{C}}(T)\).

**Proof.** (i) Let \(\lambda \in \rho_{\mathbb{C}}(T)\), then there exists an integer \(p\) such that the operator \(T^p\) is semi-Fredholm.

From the Kato decomposition, there exists \(\delta > 0\) such that

\[
\begin{align*}
\sigma_{\mathbb{C}}(T) & \subseteq (\sigma(T) \setminus \mathbb{C}) \\
\text{ind}(T) & \leq 0
\end{align*}
\]

Since \(T\) has the single-valued extension property, Proposition 1 implies that

\[
\begin{align*}
\sigma_{\mathbb{C}}(T) & \subseteq (\sigma(T) \setminus \mathbb{C}) \\
\text{ind}(T) & \leq 0
\end{align*}
\]

Therefore one verify that

\[
\begin{align*}
\sigma_{\mathbb{C}}(T) & \subseteq (\sigma(T) \setminus \mathbb{C}) \\
\text{ind}(T) & \leq 0
\end{align*}
\]

holding for \(0 \leq \lambda \leq \delta\).

Thus, by the continuity of the index, \(\text{ind}(T) \leq 0\).

(ii) This is included in part (i) since \(\text{ind}(T) = \text{ind}(T)\).

An operator \(T \in \mathcal{L}(X)\) is said to be Weyl if it is Fredholm of index zero and Browder if it is Fredholm of finite ascent and descent. The essentiel spectrum \(\sigma_e(T)\), the Weyl spectrum \(\sigma_w(T)\) and the Browder spectrum \(\sigma_B(T)\) of \(T\) are defined by

\[
\begin{align*}
\sigma_e(T) & = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm} \} \\
\sigma_w(T) & = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is Fredholm} \} \\
\sigma_B(T) & = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is Browder} \}
\end{align*}
\]

It is well known that

An operator \(T \in \mathcal{L}(X)\) is called B-Weyl if it is B-Fredholm of index zero. The B-Weyl spectrum \(\sigma_{BW}(T)\) of \(T\) is defined by
For a subset $K$ of $C$, we shall write $\text{iso}(K)$ for its isolated points. A complex number $\lambda_0$ is said to be Riesz point of $T$ in $L(X)$ if $\lambda_0 \in \sigma(T)$ and the spectral projection corresponding to the set $\{\lambda_0\}$ has finite-dimensional range. The set of all Riesz points of $T$ will be denoted by $\Pi_0(T)$. It is known that if $T \in L(X)$ and $\lambda \in \sigma(T)$ then $\lambda \in \Pi_0(T)$ if and only if $T - \lambda I$ is Fredholm of finite ascent and descent (see [19]). Consequently $\Pi(T)$.

Let $\Pi(T)$ denote the set of all poles of the resolvent of $T$ and $E_\nu(T)$ denote the set $\{ \{ a \in A : a \notin \text{Drazin invertible} \} \}$. For a normal operator $T$ acting on a Hilbert space $H$, Berkani [2, Theorem 4.5] showed that $\text{iso} A \cap \sigma(T)$, where $E(T)$ is the set of all eigenvalues of $T$ which are isolated in $\sigma(T)$. This result gives a generalization of the classical Weyl’s theorem.

II. SVEP AND GENERALIZED WEYL’S THEOREM

The concept of Drazin invertibility plays an important role for the class of B-Fredholm operators. From [12] we recall that, for an algebra $A$ with unit 1 we say that an element $a \in A$ is Drazin invertible of degree $k$ if there is an element $b$ of $A$ such that $a = b^k a$. The drazin spectrum of $a \in A$ is defined by $\text{Drazin spectrum} = \{ \lambda : \lambda \in \sigma_a \}$.

In the case of $A = L(X)$, it is well known that $T$ is Drazin invertible if and only if it has a finite ascent and descent which is also equivalent to the fact that $T = T_0 \oplus T_1$ where $T_0$ is an invertible operator and $T_1$ is a nilpotent one, see for instance [12, Proposition 6] and [7, Corollary 2.2].

Recall that $\text{iso} \sigma_a$ where $K(X)$ is the class of all compact operators acting on $X$.

It was proved in [2, Theorem 4.3] that for $T \in L(X)$, $T \in L(X)$, we will say that:

(i) $T$ satisfies Weyl’s theorem if $\text{iso} \sigma_a$.
(ii) $T$ satisfies generalized Weyl’s theorem if $\text{iso} \sigma_a$.
(iii) $T$ satisfies Browder’s theorem if $\text{iso} \sigma_a$.
(iv) $T$ satisfies generalized Browder’s theorem if $\text{iso} \sigma_a$.

Recall from [5] that if $T \in L(X)$ satisfies generalized Weyl’s theorem then it also satisfies Weyl’s theorem and if $T$ satisfies generalized Browder’s theorem then it satisfies Browder’s theorem.

We now turn to an another extension of the characterization of operators obeying Weyl’s theorem ([1, Theorem 4]).

**Theorem 3** [4, Theorem 2.5] If $T \in L(X)$ then we have

(i) $\text{iso} \sigma_a$ if and only if $\text{iso} \sigma_a$.
(ii) $\text{iso} \sigma_a$ if and only if $\text{iso} \sigma_a$.

From this theorem we obtain immediately the following corollary.

**Corollary 4** Let $T \in L(X)$, then $T$ satisfies generalized Weyl’s theorem if and only if $\text{iso} \sigma_a$ and $\text{iso} \sigma_a$.
In [15, Theorem 2.1] it is proved that if either an operator $T$ on an infinite dimensional separable Hilbert space or its Hilbert adjoint has the single-valued extension property, then the spectral mapping theorem holds for B-Weyl spectrum. Using a standard argument and the Riesz functional calculus, we obtain the same result for operators on infinite dimensional Banach spaces with a simple and short proof.

**Proposition 5** Let $T \in \mathcal{L}(X)$, then $\sigma_{\text{BW}}(f(T)) = \sigma_{\text{BW}}(f(T))$ for every $f \in \text{H}(\sigma(T))$.

**Proof.** Let $\lambda \in \sigma_{\text{BW}}(f(T))$, then $f(T) - \lambda I$ is not a B-Weyl’s operator. As there exists $\mu \in \sigma(T)$ such that $\lambda = f(\mu)$.

We have $\lambda \in \sigma(T)$ where $g$ is a non vanishing analytic function on $\sigma(T)$. So $f(\lambda) = f(f(\mu)) = f(\lambda)$.

Since $f(T) - \lambda I$ is not a B-Weyl operator, and

there exists $\beta \in \sigma_{\text{BM}}(T)$ such that $T - \beta I$ is not a B-Weyl operator and since $f(\beta) = \lambda$ we get $\beta \in \sigma_{\text{BM}}(T)$.

The opposite inclusion does not hold in general. Furthermore if $f$ is injective on $\sigma_{\text{BW}}(T)$, the last inclusion becomes an equality.

The proof of the next result is similar to that one involving $\sigma_{\text{BM}}(T)$ (see [14, Theorem 3]).

**Theorem 6** Let $T \in \mathcal{L}(X)$, if $f \in \text{H}(\sigma(T))$ is injective on $\sigma_{\text{BW}}(T)$ then $\sigma_{\text{BW}}(f(T)) = \sigma_{\text{BW}}(f(T))$.

Let $\mathcal{B} \mathcal{W}(X)$ be the class of $T \in \mathcal{L}(X)$ such that $\text{ind}(T - \lambda I) < \infty$ for all $\lambda \in \rho_{\text{BW}}(T)$ or $\text{ind}(T - \lambda I) = \infty$ for all $\lambda \in \rho_{\text{BW}}(T)$.

We recall that hyponormals operators on a Hilbert space $H$ lie in $\mathcal{B} \mathcal{W}(X)$.

The following result shows that, for operators lying in the class $\mathcal{B} \mathcal{W}(X)$, the spectral mapping theorem for complex polynomials implies the spectral mapping one for complex-valued analytic functions.

**Theorem 7** For $T \in \mathcal{L}(X)$ verifying the single-valued extension property, the following assertions are equivalent:

(i) $T \in \mathcal{B} \mathcal{W}(X)$,

(ii) $\mathcal{H}(f(T)) = \mathcal{H}(f(T))$ for all $f \in \text{H}(\sigma(T))$.

(iii) $\mathcal{H}(f(T)) = \mathcal{H}(f(T))$ for all complex polynomial $p$.

**Proof.** (i)$\Rightarrow$(ii) [22, Théorème 2.2.4] implies that $\mathcal{H}(f(T)) = \mathcal{H}(f(T))$ for all $f \in \text{H}(\sigma(T))$.

(ii)$\Rightarrow$(iii) Clear.

(iii)$\Rightarrow$(i) Assume that $T \in \mathcal{B} \mathcal{W}(X)$. Then there are $\lambda, \mu$ in $\rho_{\text{BW}}(T)$ such that $\text{ind}(T - \lambda I) > 0$ and $\text{ind}(T - \mu I) < 0$. If we consider $\text{ind}(T - \lambda I) = k$ and $\text{ind}(T - \mu I) = l$ and the polynomial $p(T)$, then $p(T)$ is a B-Fredholm operator with $\text{ind}(T - p(T)) = k + l$ and the polynomial $p(T)$, then $p(T)$ is a B-Fredholm operator with $\text{ind}(T - p(T)) = k + l$ and the polynomial $p(T)$ is a contradiction.
Proposition 8 If $T \neq T'$ has the single-valued extension property, then

$$\{ T \in \mathcal{L}(X) : \text{for any } f \in \mathcal{H}(\sigma(T)), \}$$

Proof. Let $f \in \mathcal{H}(\sigma(T))$. If $T$ or $T^*$ has the SVEP, by Proposition 2, $T$ lies in BW(X) and Theorem 7 concludes the proof.

Let $T \in \mathcal{L}(X)$, the analytical core of $T$ is the subspace, $K(T)$, defined below

The quasi-nilpotent part of $T$ is the subspace

Both subspaces, will be of particular importance in what follows, they have been introduced and studied by Mbekhta (see [8–10]). In general neither $H_0(T)$ nor $K(T)$ is closed. The following facts are easy to verify;

$$\text{for every } m \in \mathbb{N}; \text{ if } x \in X \text{, then } x \in H_0(T) \text{ if and only if } Tx \in H_0(T). \text{ If } T \text{ is invertible then } H_0(T) = \{0\}.$$ 

Theorem 9 [8, Theorem 1.6] Let $T \in \mathcal{L}(X)$, the following conditions are equivalent.

(i) $\lambda$ is an isolated point of $\sigma(T)$.

(ii) $x \in K(T),$ where $\mathcal{H}(\lambda-A) \neq \{0\}$ and the direct sum is topological.

Moreover, $\lambda$ is a pole of the resolvent, $\rho(T)$, of $T$ of order $p$ if and only if $\mathcal{H}(\lambda-A)$ and $K(T)$ are.

Our next goal is to show that generalized Browder’s theorem is satisfied for $f(T)$ whenever $T$ or $T^*$ has the single-valued extension property and $f$ in $H(\sigma(T))$. The same result was showed in [6, Theorem 1.5] for the generalized a-Browder theorem. To settle our result, we use a characterization of the pole of the resolvent in terms of ascent and descent given in [13].

Remark. It is shown in [18, Theorem 4.18] that if $T$ verify the single-valued extension property, then for any analytical function on an open neighbourhood of $\sigma(T)$, $f(T)$ verify the single-valued extension property.

Theorem 10 If $T \in \mathcal{L}(X)$ or its adjoint has the single-valued extension property, then generalized Browder’s theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.

Proof. Assume that $\lambda \in \sigma(T)$, so $T - \lambda I$ is B-Weyl, hence B-Fredholm of index 0 and by [17, Theorem 1.82], $T - \lambda I$ is Kato type. Since $T$ or $T^*$ verify SVEP, [17, Corollary 2.49] implies that $\mathcal{H}(\lambda-A) = \mathcal{H}(\lambda-A)$. Then $\lambda \in \Pi(T)$ and $\mathcal{H}(\lambda-A) = \mathcal{H}(\lambda-A)$. Conversely, if $\lambda \in \Pi(T)$ then $\lambda$ is isolated in $\sigma(T)$ and by [4, Theorem 2.3], $T - \lambda I$ is B-Weyl, that is $\lambda \in \sigma_m(T)$ and $\mathcal{H}(\lambda-A) = \mathcal{H}(\lambda-A)$. Now if $f \in \mathcal{H}(\sigma(T))$, by the last remark and the fact that $f(T)$, $f(T)$ verify SVEP and consequently we obtain

\[ \mathcal{H}(\lambda-A) = \mathcal{H}(\lambda-A). \]

From this theorem we obtain immediately the following corollary.

Corollary 11 If $T \in \mathcal{L}(X)$ or its adjoint $T^*$ has the SVEP, then
On generalized Weyl’s type theorem

(i) Generalized-Weyl’s theorem holds for $T$ if and only if $\Gamma(T) = N(T)$.

(ii) Generalized-Weyl’s theorem holds for $T^*$ if and only if $\Gamma(T^*) = N(T^*)$.

The next result rewrite some results due to C. Schmoeger [13] as follows.

**Proposition 12** Let $T \in L(X)$, the following conditions are equivalent

(i) $\lambda \in \Pi(T)$.

(ii) $\lambda \in E(T)$ and there exists an integer $p \geq 1$ for which $R(T) = R(T^*)$.

(iii) $\lambda \in E(T)$ and there exists an integer $p \geq 1$ for which $K(T) = K(T^*)$.

(iv) $\lambda \in E(T)$ and $T - \lambda I$ is of finite descent.

**Proof.** Without loss of generality we can assume that $\lambda = 0$.

(i) $\Rightarrow$ (ii) Since 0 is a pole of the resolvent of $T$ of order $p$, it is an eigenvalue of $T$ and an isolated point of the spectrum of $T$. Hence $0 \in E(T)$. Finally by Theorem 9 $\Pi(T) = N(T)$.

(ii) $\Rightarrow$ (iii) If there exists $p \geq 1$ such that $\Pi(T) = N(T)$ and $0 \in E(T)$ from [8, Théorème 1.6] we have $X = \{0\}$. Then one obtain $K(T) = K(T^*)$ and since $\Pi(T) = N(T)$, $K(T) = K(T^*)$ follows that $K(T) = K(T^*)$.

(iii) $\Rightarrow$ (iv) If there exists $p \geq 1$ such that $K(T) = K(T^*)$, since $T(K(T)) = K(T^*)$ it follows that $d(T) < \infty$.

(iv) $\Rightarrow$ (i) Suppose that $0 \in E(T)$ and $d(T) < \infty$. Since 0 is isolated in $\sigma(T)$, by [13, Theorem 4] $X = \{0\}$ and $H(T) \cong \{0\}$ is closed. Hence by [13, Theorem 2(b)] $T$ has the SVEP at 0 and finally [13, Theorem 5] gives $0 \in \Pi(T)$.

The following theorem follows immediately from Corollary 11 and Proposition 12.

**Theorem 13** Let $T \in L(X)$ such that $T$ or its adjoint $T^*$ has the single-valued extension property then the following conditions are equivalent:

(i) Generalized-Weyl’s theorem holds for $T$.

(ii) $\forall \lambda \in E(T)$ there exists $p \geq 1$ for which $K(T) = K(T^*)$.

(iii) $\forall \lambda \in E(T)$ there exists $p \geq 1$ for which $K(T) = K(T^*)$.

(iv) $\forall \lambda \in E(T)$, $T - \lambda I$ is of finite descent.

**REFERENCES**

**Journal Papers:**


On generalized Weyl’s type theorem


Books:

Theses: