Periodic Solutions of abstract neutral functional differential equations

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Abstract

We characterize the existence of periodic solutions for a class of abstract neutral functional differential equations described in the form:

\[ \frac{d}{dt} x(t) = A[x(t) - Bx(t - r)] + L(x_t) + f(t), t \in \mathbb{R} \] (1)

Keywords: functional differential equations

1. Introduction:

Let \( X \) be a Banach space endowed with a norm \( ||| \cdot ||| \) and \( r \) be non negative real number.

The main objective of this paper is to study the existence of periodic solutions for the class of linear abstract neutral differential equations (1):

\[ C = C([-r,0]; X) \] be the Banach space of continuous functions mapping the interval \([-r,0]\) into \( X \). The function \( x_t \) given by \( x_t(\theta) = x(t+\theta) \) for \( \theta \) in appropriate domain, denotes the segment or the "history" of the function \( x(\cdot) \) at \( t \).

\( L \) is a bounded linear map defined on an appropriate space, and \( f : \mathbb{R} \rightarrow X \) is a locally \( p \)-integrable and \( 2\pi \)-periodic function for \( 1 \leq p < +\infty \).

We assume that \( A : D(A) \subseteq X \rightarrow X \) and \( B \subseteq X \rightarrow X \) are closed linear operators.

We denote

\[ H^{1,p}(T; X) = \{ u \in L^p(T; X) : \exists v \in L^p(T; X), \hat{v}(k) = ik\hat{u}(k) \text{ for all } k \in \mathbb{Z} \} \]
2. Preliminaries:

We denote by $T$ the group defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. There is an obvious identification between functions on $T$ and $2\pi$-periodic functions on $\mathbb{R}$. We consider the interval $[0, 2\pi)$ as a model for $T$.

For a function $f \in L^1(T; X)$, we denote by $\hat{f}(k)$, $k \in \mathbb{Z}$ the $k$-th Fourier coefficient of $f$:

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t) \, dt \quad \text{for } k \in \mathbb{Z} \text{ and } t \in \mathbb{R}.$$  

Denote $f_{\tau}(t) := f(t+\tau)$, $\tau \in \mathbb{Z}$; then it the follows from the definition that $\hat{f}_{\tau}(k) = e^{ik\tau} \hat{f}(k)$, $\tau \in T$.

Let $f \in L^p(T, X)$. Then by Fefer's theorem, one has

$$f = \lim_{n \to \infty} \sigma_n(f)$$

in $L^p(T, X)$ where

$$\sigma_n(f) := \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \hat{f}(k)$$

with $e_k(t) := e^{ikt}$

A Banach space $X$ is said to be UMD, if the Hilbert transform is bounded on $L^p(R, X)$ for all $p \in (1, \infty)$.

Definition 1: Let $X$ and $Y$ be a Banach spaces. A family of operators $T \subset B(X, Y)$ is called $R$-bounded, if there is a constant $C > 0$ and $p \in [1, \infty)$ such that for each $N \in \mathbb{N}, T_j \in T$, $x_j \in X$ and for all independent, symmetric, $\{-1, 1\}$-valued random variables $r_j$ on a probability space $(\Omega, M, \mu)$ the inequality

$$\left\| \sum_{j=1}^{N} r_j T_j x_j \right\|_{L^p(\Omega, Y)} \leq C \left\| \sum_{j=1}^{N} r_j x_j \right\|_{L^p(\Omega, Y)}$$

is valid. The smallest such $C$ is called $R$-bounded of $T$, we denot it by $R_p(T)$. 

www.iosrjournals.org 87 | Page
Definition 2: For $1 \leq p \leq \infty$ we say that a sequence $\{M_k\}_{k \in \mathbb{Z}} \subset B(X,Y)$ is an $L^p$-multiplier if, for each $f \in L^p(T,X)$, there exists $u \in L^p(T,Y)$ such that

$$\hat{u}(k) = M_k \hat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$ 

Theorem 1: Let $X$, $Y$ be UMD space and let $\{M_k\}_{k \in \mathbb{Z}} \subset B(X,Y)$. If the sets $\{M_k\}_{k \in \mathbb{Z}}$ and $\{k(M_{k+1} - M_k)\}_{k \in \mathbb{Z}}$ are $R$-bounded, then $\{M_k\}_{k \in \mathbb{Z}}$ is an $L^p$-multiplier for $1 < p < \infty$.

3. A Criterion for Periodic Solutions:

We consider $\Delta_k = ikI - ikB_k - A(I - B_k) - L_k$, for all $k \in \mathbb{Z}$.

Denote by $B_k := \exp^{-ikr}B$, $L_k(x) := L(e^{ikg}x)$ and $e_k(t) := e^{ikt}$ for all $k \in \mathbb{Z}$ and $\sigma_Z(\Delta) = \{k \in \mathbb{Z} : \Delta_k \text{ has no inverse}\}$

And we define $D_k = (ikI - A(I - B_k) - L_k)^{-1}$

3.1. Existence of Strong Solution:

Definition 3: Let $A$ be a closed linear operator on $X$. A function $x(.)$ solution of the problem (1) if $x \in H^{1,p}(T;X) \cap L^p(T;X)$ and (1) holds for almost all $t \in [0, 2\pi]$. 

Theorem 2: Let $X$ be a Banach space and $1 < p < +\infty$. Suppose that for every $f \in L^p(T,X)$ there exists a unique strong solution of Eq (1). Then

1. for every $k \in \mathbb{Z}$ the operator $(ikI - A(I - B_k) - L_k)$ has bounded inverse

2. The set is $R$-bounded and $\{ikD_k\}_{k \in \mathbb{Z}}$ is $R$-bounded.

Lemma 1: [2, Lemma 4.2]

Let $u \in C(T,X)$. Then

$$L(X_u(k)) = L_k \hat{x}(k).$$

Proof of Theorem 2:

1) Let $k \in \mathbb{Z}, y \in X$

for $f(t) = e^{ikr}y$, $\exists x \in H^{1,p}(T,X)$ such that:

$$\frac{dx}{dt}(t) = A(x(t) - Bx(t-r)) + L(x_t) + f(t)$$

Taking Fourier transform, $L$ is linear and bounded, we obtain

$$ik\hat{x}(k) = A(I - B_k)\hat{x}(k) + L_k\hat{x}(k) + \hat{f}(k)$$

$$(ikI - A(I - B_k) - L_k)\hat{x}(k) = \hat{f}(k) = y$$

$(ikI - A(I - B_k) - L_k)$ is surjective.

Let $x \in \text{Ker}((ik - A(I - B_k) - L_k))$, that is $A(I - B_k)x + L_kx = ikx$, then $u(t) = e^{ikx}$ defines a periodic solution of (1) corresponding to the function $\tilde{f}(t) = 0$. Consequently, $u(t) = 0$ and $x = 0$.

2) let $f \in L^p(T,X)$. By hypothesis, there exists a unique $x \in H^{1,p}(T,X)$ such that (1) equation is valid. Taking Fourier transforms, we deduce that $(ikI - A(I - B_k) - L_k)\hat{x}(k) = \hat{f}(k)$ for all $k \in \mathbb{Z}$. Hence

$$ik\hat{x}(k) = ik(ikI - A(I - B_k) - L_k)^{-1}\hat{f}(k)$$

for all $k \in \mathbb{Z}$.

On the other hand, since $x \in H^{1,p}(T,X)$, there exists $v \in L^p(T,X)$ such that

$$\hat{v}(k) = ik\hat{x}(k).$$

This proves claim.
3.2. Existence of weak solution:

**Definition 4**: Let \( A \) be a closed linear operator on \( X \). A function \( x(.) \) is called a weak solution of the problem (1) if:
\[
\int_0^1 (x(s) - Bx(s)) ds \in D(A) \quad \text{and} \quad x(t) - x(0) = A \int_0^t (x(s) - Bx(s)) ds + \int_0^t (Lx_s + f(s)) ds, \quad 0 \leq t \leq 2\pi.
\]

**Theorem 3**: Let \( f \in L^p(T, X) \), Assume that \( D(A) = X \); if \( x(.) \) is said to be a weak solution of Eq (1) then \((ikI - A(I-B_k) - L_k)\hat{x}(k) = \hat{f}(k)\) for all \( k \in \mathbb{Z} \).

**proof**: \( x(.) \) is a weak solution of Eq (1) then
\[
x(t) - x(0) = A \int_0^t x(s) ds + \int_0^t (Gx_s + f(s)) ds
\]
\[
t = 2\pi
\]
\[
x(2\pi) - x(0) = A \int_0^{2\pi} (x(s) - Bx(s)) ds + \int_0^{2\pi} (Lx_s + f(s)) ds; \quad \text{or} \quad x(2\pi) = x(0)
\]
then
\[
A \int_0^{2\pi} (x(s) - Bx(s)) ds + \int_0^{2\pi} (Lx_s + f(s)) ds = 0
\]
\[
(AI - B_0 + L_0)\hat{x}(0) + \hat{f}(0) = 0
\]
\[
(0AI - B_0 - L_0)\hat{x}(0) = \hat{f}(0) \quad \text{which shows that the assertion holds for} \quad k = 0.
\]
Define \( v(t) = \int_0^t (x(s) - Bx(s)) ds \)
And \( g(t) = x(t) - x(0) - \int_0^t (Lx_s + f(s)) ds \)
by lemma 3.1 [2]
We have \( \hat{v}(k) = \frac{i}{k} (\hat{x}(0) - B\hat{x}(0)) - \frac{i}{k} (\hat{x}(k) - B\hat{x}(k)) \) (remark 2.3 [2])
\[
\hat{g}(k) = \hat{x}(k) - \frac{i}{k} L_0 \hat{x}(0) - \frac{i}{k} L_k \hat{x}(k) - [\frac{i}{k} \hat{f}(0) - \frac{i}{k} \hat{f}(k)]
\]
\[
\hat{g}(k) = \hat{x}(k) - \frac{i}{k} L_0 \hat{x}(0) + \frac{i}{k} L_k \hat{x}(k) - \frac{i}{k} \hat{f}(0) + \frac{i}{k} \hat{f}(k)
\]
\[
A \hat{v}(k) = \frac{i}{k} A(I - B_0) \hat{x}(0) - \frac{i}{k} A(I - B_k) \hat{x}(k)
\]
Then
\[
\text{ik}_k\dot{x}(k) + L_0\dot{x}(0) - L_k\dot{x}(k) + f(0) - \dot{f}(k) = -A(I-B_0)\dot{x}(0) + A(I-B_k)\dot{x}(k)
\]
\[\Leftrightarrow [ \text{ik}_k\dot{x}(k) - A(I-B_k)\dot{x}(k) - L_k\dot{x}(k) - \dot{f}(k)] - [A(I-B_0)\dot{x}(0) + L_0\dot{x}(0) + \dot{f}(0)] = 0
\]
\[\Leftrightarrow \text{ik}_k\dot{x}(k) - A(I-B_k)\dot{x}(k) - L_k\dot{x}(k) - \dot{f}(k) = 0
\]
\[\Leftrightarrow \text{ik}_k\dot{x}(k) - A(I-B_k)\dot{x}(k) - L_k\dot{x}(k) = \dot{f}(k).
\]

**Theorem 4** Let \( f \in L^p(T,X) \), Assume that \( \overline{D(A)} = X \); if \( x(\cdot) \) is said to be a weak solution of Eq (2) and \( (ikI - A(I - B_k) - L_k) \) has a bounded inverse. Then \( (ikI - A(I - B_k) - L_k)^{-1} \) is an \( L^p \)-multiplier.

Proof: from theorem (1) we have \( \dot{x}(k) = (ikI - A(I - B_k) - L_k)^{-1}\dot{f}(k) \), for all \( f \in L^p(T,X) \)

**Main result:**

Our main result in this paper, establish that the converse of theorem (2) and the give the definition of Mild solution

**Theorem 5:**

Let \( X \) be a UMD space and let \( A : D(A) \subset X \rightarrow X \) be a closed linear operator. The following assertions are equivalent for \( 1<p<\infty \).

1. for every \( f \in L^p(T,X) \) there exists a unique strong solution of Eq (1)

2. for every \( k \in Z \) the operator \( (ikI - A(I-B_k) - L_k) \) has bounded inverse and the set is \( R \)-bounded and \( \{ikD_k\}_{k \in Z} \) is \( R \)-bounded.
proof:

\(1\Rightarrow 2\) Let \(f \in L^p(T, X)\). Define \(D_k = (ikI - A(I - B_k) - L_k)^{-1}\), the family \(\{ikD_k\}_{k \in \mathbb{Z}}\) is an \(L^p\)-multiplier it is equivalent to the family \(\{D_k\}_{k \in \mathbb{Z}}\) is an \(L^p\)-multiplier that maps \(L^p(T, X)\) into \(H^{1,p}(T, X)\), i.e. there exists \(x \in H^{1,p}(T, x)\) such that

\[
\hat{x}(k) = D_k \hat{f}(k) = (ikI - A(I - B_k) - L_k)^{-1} \hat{f}(k)
\]

(1.1)

In particular, \(x \in L^p(T, X)\) and there exists \(v \in L^p(T, X)\) such that

\[
\hat{v}(k) := \hat{v}(k) = ik \hat{x}(k)
\]

By Fejer’s theorem one has in \(L^p([-\tau_2, 0], X)\)

\[
x_t(\theta) = x(t+\theta) = \lim_{n \to +\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} e^{ik\theta} \hat{x}(k)
\]

Hence in \(L^p(T, X)\) we obtain

\[
x_t = \lim_{n \to +\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} e_k \hat{x}(k)
\]

Then, since \(L\) is linear and bounded

\[
Lx_t = \lim_{n \to +\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} L(e_k \hat{x}(k))
\]

\[
= \lim_{n \to +\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} L_k \hat{x}(k)
\]

By (1.1) and (1.2) we have

\[
\hat{x'}(k) = ik \hat{x}(k) = A(I-B_k) \hat{x}(k) + L_k \hat{x}(k) + \hat{f}(k). \quad \text{for all } k \in \mathbb{Z}.
\]

Then using that \(A\) and \(B\) are closed we conclude that \((x(t)-Bx(t-r))\in D(A)\), and from the uniqueness theorem of Fourier coefficients, that equation (2) is valid for \(t \in T\). [3. lemma 3.1]
Definition 5 : of Mild solution about convert of weak solution

Introduction :

Assume that \( A \) generates a \( C_0 \)-semigroup \( T(.) \) on \( X \); and \( x(.) \) is a weak solution, then we have

\[
x(t) - x(0) = A \int_0^t (x(s) - Bx(s-r)) \, ds + \int_0^t (Gx_s + f(s)) \, ds
\]

\[
\int_0^t T(t-s) (x(s) - x(0)) \, ds = \int_0^t T(t-s) A \int_0^s (x(\xi) - Bx(\xi-r)) \, d\xi \, ds + \int_0^t T(t-s) \int_0^s (L(x_\xi) + f(\xi)) \, d\xi \, ds
\]

\[
= \int_0^t (T(t-s)-I)(x(s) - Bx(s-r)) \, ds + \int_0^t T(t-s) \int_0^s (L(x_\xi) + f(\xi)) \, d\xi \, ds
\]

Then

\[
\int_0^t T(t-s) (Bx(s-r) - x(0)) \, ds = - \int_0^t (x(s) - Bx(s-r)) \, ds + \int_0^t T(t-s) \int_0^s (L(x_\xi) + f(\xi)) \, d\xi \, ds
\]

\[
\int_0^t (x(s) - Bx(s-r)) \, ds + \int_0^t T(t-s) (Bx(s-r) - x(0)) \, ds = \int_0^t T(t-s) \int_0^s (L(x_\xi) + f(\xi)) \, d\xi \, ds
\]

\[
A \int_0^t (x(s) - Bx(s-r)) \, ds + A \int_0^t T(t-s) (Bx(s-r) - x(0)) \, ds = A \int_0^t T(t-s) \int_0^s (L(x_\xi) + f(\xi)) \, d\xi \, ds
\]

\[
A \int_0^t (x(s) - Bx(s-r)) \, ds + A \int_0^t T(t-s) (Bx(s-r) - x(0)) \, ds = \int_0^t (T(t-s)-I) \int_0^s (L(x_\xi) + f(\xi)) \, d\xi \, ds
\]

or \( x(.) \) is a weak solution then

\[
x(t) - x(0) = A \int_0^t T(t-s) (x(0) - Bx(s-r)) \, ds + \int_0^t T(t-s) (L(x_s) + f(s)) \, ds
\]

Our object, establish the converse of this result

Definition 6 : Assume that \( A \) generates a \( C_0 \)-semigroup \( T(.) \) on \( X \). A function \( x(.) \) is called a mild solution of the problem (1) if :

\[
\int_0^t T(t-s) (x(0) - Bx(s-r)) \, ds \in D(A) \quad \text{and}
\]

\[
x(t) - x(0) = A \int_0^t T(t-s) (x(0) - Bx(s-r)) \, ds + \int_0^t T(t-s) (L(x_s) + f(s)) \, ds \quad 0 \leq t \leq 2\pi.
\]
Corollary 1  Assume that A generates a $C_0$-semigroup $T(\cdot)$ on $X$; let $f \in L^p(T,X)$

$x(\cdot)$ is a weak solution $\iff$ $x(\cdot)$ is a mild solution

Proof:

$\Rightarrow$ by introduction

$\Leftarrow$ suppose that $x(\cdot)$ is a mild solution of Eq (2) then

$$x(t) - x(0) = A \int_0^t T(t-s) (x(0) - Bx(s-r)) ds + \int_0^t T(t-s) (L(x_s) + f(s)) ds$$

$$\int_0^t (x(s) - x(0)) ds = \int_0^t A \int_0^s T(t-\xi) (x(0) - Bx(\xi-r)) d\xi ds + \int_0^t \int_0^s T(t-\xi) (L(x_\xi) + f(\xi)) d\xi ds$$

$$\int_0^t (x(s) - x(0)) ds = \int_0^t (T(t-s)-I) (x(0) - Bx(s-r)) ds + \int_0^t \int_0^s T(t-\xi) (L(x_\xi) + f(\xi)) d\xi ds$$

$$A \int_0^t (x(s) - x(0)) ds = A \int_0^t (T(t-s)-I) (x(0) - Bx(s-r)) ds + \int_0^t (T(t-s)-I) (L(x_s) + f(s)) ds$$

$$A \int_0^t (x(s) - x(0)) ds + \int_0^t (L(x_s) + f(s)) ds + A \int_0^t (x(0) - Bx(s-r)) ds = A \int_0^t T(t-s) (x(0) - Bx(s-r)) ds + \int_0^t (T(t-s)-I) (L(x_s) + f(s)) ds$$

$$A \int_0^t (T(t-s) (x(0) - Bx(s-r)) ds + \int_0^t (T(t-s)-I) (L(x_s) + f(s)) ds = A \int_0^t (x(s) - x(0)) ds + \int_0^t (L(x_s) + f(s)) ds$$

$$x(t) - x(0) = A \int_0^t (x(s) - Bx(s-r)) ds + \int_0^t (L(x_s) + f(s)) ds$$

Then $x(\cdot)$ is a weak solution.

Proposition 1: Assume that A generates a $C_0$-semigroup $T(\cdot)$ on X; if

$(ikI - A(I - B_k) - L_k)^{-1}$ is an $L^p$-multiplier Then there exists a unique weak (mild) solution of Eq (1).
proof: let \( f \in L^p(T,X) \), then 
\[
 f(t) = \lim_{n \to +\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} \hat{f}(k)
\]
or 
\[
 (ikI - A(I - B_k) - L_k)^{-1}
\]
is an \( L^p \)-multiplier then there exists \( x \in L^p(T,X) \) such that 
\[
 \hat{x}(k) = (ikI - A(I - B_k) - L_k)^{-1} \hat{f}(k)
\]
put 
\[
 x_n(t) = \lim_{n \to +\infty} \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e^{ikt} (ikI - A(I - B_k) - L_k)^{-1} \hat{f}(k)
\]
then 
\[
 x_n(t) \to x(t) \text{ and } x_n \text{ is strong } L^p \text{-solution of Eq (1) and } x_n \text{ verified}
\]

\[
x_n(t) - x_n(0) = A \int_{0}^{t} ((x_n(t-s)) - Bx_n(t-s))ds + \int_{0}^{t} (G((x_n)_s) + f_n(s))ds
\]

we put 
\[
y_n = x_n(0)
\]

\[
x_n(t) = y_n + A \int_{0}^{t} ((x_n(t-s)) - Bx_n(t-s))ds + \int_{0}^{t} (L((x_n)_s) + f(s))ds
\]

\[
t = 2\pi
\]

\[
x_n(2\pi) = y_n + A \int_{0}^{2\pi} ((x_n(t-s)) - Bx_n(t-s))ds + \int_{0}^{2\pi} (L((x_n)_s) + f(s))ds
\]

\[
(n \to \infty)
\]

\[
 y = y + A \int_{0}^{2\pi} ((x(s)) - Bx(2\pi-r))ds + \int_{0}^{2\pi} (L(x_s) + f(s))ds
\]

\[
x(t) = y + A \int_{0}^{t} (x(s)) - Bx(t-s))ds + \int_{0}^{t} (L(x_s) + f(s))ds =: g(t)
\]

\[
x(2\pi) = g(2\pi) = y + A \int_{0}^{2\pi} ((x(s)) - Bx(2\pi-r))ds + \int_{0}^{2\pi} (L(x_s) + f(s))ds
\]

\[
y = g(0)
\]

\[
 \Rightarrow x(2\pi) = x(0)
\]
we conclude that \( x(\cdot) \) is a \( 2\pi \)-periodic weak (mild) solution of Eq (1).

### 4 Exemple:

\[
\frac{d}{dt} x(t) = A(x(t) - Bx(t-r)) + Lx_t + f(t)
\]

let \( A \) be a closed linear operator and \( X \) be a UMD space, and

\[
\sup_k \| (ikI - A(I - B_k))^{-1} \| = : M < \infty \text{ and } \| L \| < \frac{1}{r_2^{1/2}} r_2^{1/2} \text{ then Eq (1) has a unique weak solution.}
\]

we have 
\[
 ikI - A(I - B_k) - L_k = [ikI - A(I - B_k)][I - L_k(ikI - A(I - B_k))^{-1}]
\]
it follows that \( ikI - A(I - B_k) - L_k \) is invertible whenever 
\[
 \| L_k(ikI - A(I - B_k))^{-1} \| < 1 \text{ [7. Theorem 1.17]}
\]
observe that 
\[
 \| L_k \| \leq r_2^{1/2} \| L \|
\]
Hence 
\[
 \| L_k(ikI - A(I - B_k))^{-1} \| \leq r_2^{1/2} \| L \| M = : \alpha
\]
Therefore, under the condition 
\[
 \| L \| < \frac{1}{r_2^{1/2} M}
\]

\[
 (ikI - A(I - B_k) - L_k)^{-1} = [ikI - A(B_k)]^{-1} [I - L_k(ikI - A(I - B_k))^{-1}]
\]

\[
 = [ikI - A(B_k)]^{-1} \sum_{n=0}^{\infty} [L_k (ikI - A(I - B_k))^{-1}]^n
\]
it follows that 
\[
 \| ik(ikI - A(I - B_k))^{-1} \| \leq \| ik(ikI - A(I - B_k))^{-1} \| \sum_{n=0}^{\infty} \alpha^n
\]

\[
 \leq \frac{M+1}{1-\alpha} \text{ then } ikD_k \text{ is } R\text{-bounded.}
Bibliographie


