Solving n power class (Q) operators using MATLAB

Dr. T. Veluchamy, K.M.Manikanadan, T.Ramesh
Dept. of Mathematics, Dr.SNS Rajalakshmi college of Arts & Science ,Coimbatore-49, TamilNadu,India

Abstract: In this paper we investigate the characterisation of n power class (Q) operators on Hilbert space using MATLAB. Mathematics Subject Classification: 47B99, 47B15
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I. INTRODUCTION

Let H be a Hilbert space and L(H) is the algebra of all bounded linear operators acting on H. An operator T in L(H) is called normal if $T^*T = TT^*$, class(Q) if $T^{-2}T^2 = (T^*T)^2$, n power class (Q) if $T^{-2^n}T^{2^n} = (T^*T)^n$. In general an power class(Q)operator need not be a normal operator. Existence of Operators on 2 power class (Q) and 3 power class (Q) are verified by using MATLAB R2008a version.

Program no. 1

Consider the operators $S = \begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix}$ and $T = \begin{pmatrix} i & 0 \\ 1 & -i \end{pmatrix}$. These two operators are 2 power class (Q) operators on the complex Hilbert space. $S+T$ is 2 power class (Q)But $S+T$ is not normal.

$$
S = \begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix}, 
T = \begin{pmatrix} i & 0 \\ 1 & -i \end{pmatrix}
$$

Sum = $S+T$

ConjSum = conj(Sum)

ConjSumsq = ConjSum*ConjSum

Sumpow4 = Sum^4

L = ConjSumsq*Sumpow4...........(1)

Sumpow2 = Sum^2

M = ConjSum* Sumpow2...........(2)

N = M*M............(3)

k = Sum*ConjSum...........(4)

v = ConjSum*Sum

On executing this program, equations (1) and (2) are same. From the output we can verify that $(S+T)^2 (S+T)^4 = (S+T)(S+T)^2$. Hence $S+T$ is 2 power class (Q), Equations (3) and (4) are not equal.. Hence $S+T$ is not normal.

Definition:Hadamard matrices are matrices of 1’s and -1’s whose columns are orthogonal, $H^*H = n*I$ where [n n]=size(H) and I = eye(n,n).

An n×n Hadamard matrix with n > 2 exists only if rem(n,4) = 0. This function handles only the cases where n, n/12, or n/20 is a power of 2.

Examples

The command hadamard (4) produces the 4-by-4 matrix:

$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$

Program No.2

Consider Hadamard matrix of order 40. Here $n = 40$ and $\frac{40}{20} = 2^1$ (power of 2). We verify this matrix of order 40belongs to 4power Class (Q)

$t = \text{hadamard}(40)$;

tstar = t;
tstarsq= tstar*tstar;

tpow80=t^80;

L = tstarsq*tpow80 ..........(1)

tpow40=t^40;

S = tstar*tpow40;
V = S*S  

On executing this program, equations (1) and (2) are same. From the output we can verify that $T^{*2}T^{*80} = (T^*T^{40})^2$. Hence Hadamard matrix of order 40 belongs to 40 power class Q.

**Program no: 03**

Consider Hadamard matrix of order 48. Here $n = 48$ and $48^2 = 4 = 2^2$ (power of 2). We verify this matrix of order 48 belongs to 48 power class (Q).

\[
t = \text{hadamard}(48);
tstar = t;
tstarsq = tstar*tstar;
tpow96 = tstar^96;
K = tstarsq*tpow96  
\]

\[
tpow48 = tstar^48;
M = tstar*tpow48;
N = M*M
\]

On executing this program, equations (1) and (2) are same. From the output we can verify that $T^{*2}T^{*96} = (T^*T^{*48})^2$. Hence Hadamard matrix of order 40 belongs to 40 power class Q.

**Definition:**

Hilbert matrix is an $n \times n$ matrix with elements $(h_{ij})$ where $h_{ij} = \frac{1}{i+j-1}$ for $1 \leq i, j \leq n$.

**Program No 04**

Let $T = \begin{pmatrix}
1 & 1 & 1 \\
\frac{1}{2} & \frac{1}{2} & 1 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{pmatrix}$ be an operator acting on 3 dimensional Hilbert space. Then $T$ is 2 power class (Q), the matrix of $T$ is a Hilbert matrix of order 3. $T$ is 2 power class (Q) if it satisfies $T^{*2}T^{*4} = (T^*T^2)^2$. We can also verify $T$ is 2 normal if $T^*T^2 = T^2T^*$.

\[
T = [1\ 1/2\ 1/3;1/2\ 1/3\ 1/4;1/3\ 1/4\ 1/5]
\]

\[
\text{ConjT} = \text{conj}(T)
\]

\[
\text{ConjTsqs} = \text{ConjT}*\text{ConjT}
\]

\[
Tpow4 = T^4
\]

\[
L = \text{ConjTsqs}^*\text{Tpows}  
\]

\[
\text{Tpows} = T^2
\]

\[
M = \text{ConjT}*\text{Tpows}
\]

\[
N = M*M
\]

On executing this program, equations (1) and (2) are same. From the output we can verify that $T^{*2}T^{*4} = (T^*T^2)^2$.

// Hilbert matrix of order 3 is 2 normal\/

\[
O = \text{ConjT}*\text{Tpows}  
\]

\[
\text{P} = \text{Tpows}^*\text{ConjT}
\]

On executing this program, equations (3) and (4) are same. From the output we can verify that $T^2T^2 = T^2T^*$.

**Program No 05**

Let $T = \begin{pmatrix}
1 & 1 & 1 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{pmatrix}$ be an operator acting on 3 dimensional Hilbert space. Then $T$ is 3 power class (Q), the matrix of $T$ is a Hilbert matrix of order 3. $T$ is 3 power class (Q) if it satisfies $T^{*2}T^{*6} = (T^*T^3)^2$.

Hilbert matrix of order 3 is 3 power class Q

\[
T = [1\ 1/2\ 1/3;1/2\ 1/3\ 1/4;1/3\ 1/4\ 1/5]
\]

\[
\text{ConjT} = \text{conj}(T)
\]

\[
\text{ConjTsqs} = \text{ConjT}*\text{ConjT}
\]

\[
Tpow6 = T^6
\]

\[
L = \text{ConjTsqs}^*\text{Tpows}  
\]

\[
\text{Tpows} = T^3
\]

\[
M = \text{ConjT}*\text{Tpows}
\]

\[
N = M*M
\]

On executing this program, equations (3) and (4) are same. From the output we can verify that $T^2T^2 = T^2T^*$. 


On executing this program, equations (1) and (2) are same. From the output we can verify that \( T^{*2}T^{*6} = (T^{*4}T^{*2})^2 \).

**Definition:**
A bounded operator \( T \) on a Hilbert space is said to be nilpotent if \( T^n = 0 \) for some \( n \).

**Program No 6**
Consider the following irreducible nilpotent operators acting on \( C_2 \).

\[
R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}
\]

Then both \( S \) and \( T \) are normal and \( \in \mathbb{C}^2 \) power class (Q).

\[
R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{Conj}R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}
\]

**Program No 7**
Consider the operator \( T = \begin{bmatrix} -i & 0 \\ 2 & i \end{bmatrix} \) acting on 2 dimensional complex Hilbert space which is \( 2 \) power class (Q) but not \( 3 \) power class (Q).

\[
T = \begin{bmatrix} -i & 0 \\ 2 & i \end{bmatrix}, \quad \text{Conj}T = \begin{bmatrix} -i & 0 \\ 2 & i \end{bmatrix}
\]

On executing this program, equations (1) and (2) are same. From the output we can verify that \( RR^* = R^* R \).

Hence \( R \) is normal. Similarly, On executing this program, equations (3) and (4) are same. From the output we can verify that \( R^{*2}R^4 = (R^*R^2)^2 \) The proof for \( S \) is similar.

**Program No 8**
Consider the operators \( T = \begin{bmatrix} i & 1 \\ 0 & -i \end{bmatrix}, \quad S = \begin{bmatrix} i & 0 \\ 1 & -i \end{bmatrix} \) acting on 2 dimensional complex.

Hilbert space. Then the Product \( ST \) of \( 2 \) power Class Q operators need not be \( 2 \) power class Q operator.

\[
S = \begin{bmatrix} i,1,0 & -i \\ i,0,1 & -i \end{bmatrix}
\]

\[
T = \begin{bmatrix} i,0,1 & -i \\ i,1,0 & -i \end{bmatrix}, \quad \text{Product} = ST
\]

On executing this program, equations (1) and (2) are same. From the output we can verify that \( T^{*2}T^{*6} = (T^{*4}T^{*2})^2 \). Hence \( T \) is \( 2 \) power class (Q). Equations (3) and (4) are not same. From the output we can verify that \( T^{*3}T^{*6} = (T^{*4}T^{*2})^2 \). Hence \( T \) is not \( 3 \) power class (Q).
Q = V*V ........(2)
On executing this program, equations (1) and (2) are not same. From the output we can verify that, $(ST)^4 \neq (ST)^2 (ST)^2$.

References