A Note on Generalized Weighted Arithmetic Mean Summability Factors via Quasi-B-Power Increasing Sequence

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Abstract: In this paper we have established a theorem on generalized summability factors via quasi-β-power increasing sequence, which gives some new results and generalizes some previous known results.

Keywords: Weighted arithmetic mean summability, summability, summability factors and quasi-β-power increasing sequence.

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I. Introduction:

Let \( \sum a_n \) be a given infinite series with partial sums \( \{ s_n \} \). We denote by \( u_n^\alpha \) and \( t_n^\alpha \) the \( n \)-th Cesaro means of order \( \alpha \), with \( \alpha > -1 \), of the sequence \( (s_n) \) and \( (na_n) \) respectively such that

\[
\begin{align*}
u_n^\alpha &= \frac{1}{A_n^\alpha} \sum_{v=0}^{n-1} A_{n-v}^\alpha s_v \\
t_n^\alpha &= \frac{1}{A_n^\alpha} \sum_{v=0}^{n-1} A_{n-v}^\alpha va_v
\end{align*}
\]

(1.1)

(1.2)

where \( A_n^\alpha = O(n^\alpha) \), \( \alpha > -1 \), \( A_0^\alpha = 1 \) and \( A_n^\alpha = 0 \) for \( n > 0 \).

A series \( \sum a_n \) is said to be summable \( |C, \alpha|_k, k \geq 1 \) if (FLETT [7]).

\[
\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} \frac{|t_n^\alpha|^k}{n} < \infty
\]

(1.3)

and \( \sum a_n \) is said to be summable \( |C, \alpha, \delta|_k, k \geq 1 \) and \( \delta \geq 0 \) if (FLETT [7]).

\[
\sum_{n=1}^{\infty} n^{\delta k-1} |u_n^\alpha|^k < \infty
\]

(1.4)

Let \( (p_n) \) be a sequence of positive numbers such that

\[
P_n = \sum_{v=0}^{n} p_v \to \infty \text{ as } n \to \infty (P_{-1} = p_{-1} = 0, \ i \geq 1)
\]

(1.5)

The sequence to sequence transformation

\[
\sigma_n = \frac{1}{P_n} \sum p_vs_v
\]

(1.6)

defines the sequence \( (\sigma_n) \) of the Weighted arithmetic mean or simply \( (\bar{N}, p_n) \) mean of the sequence \( (s_n) \) generated by the sequence of coefficients \( (p_n) \) (HARDY [8]). The series \( \sum a_n \) is said to be summable \( |\bar{N}, p_n|_k, k \geq 1 \) if (BOR [7])

\[
\sum_{n=1}^{\infty} \left( \frac{p_n}{p_{-1}} \right)^{k-1} |\Delta \sigma_{n-1}|^k < \infty
\]

(1.7)

and it is said to be summable \( |\bar{N}, p_n, \delta|_k; k \geq 1 \) and \( \delta \geq 0 \) if (BOR [3])
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\[
\sum_{n=1}^{\infty} \left( \frac{P_n}{p_n} \right)^{\delta k + 1} \left| \Delta \sigma_{n-1} \right|^k < \infty
\]  

where \( \Delta \sigma_{n-1} = \sigma_n - \sigma_{n-1} = \frac{P_n}{p_n p_{n-1}} \sum_{v=1}^{n} a_v ; n \geq 1 \).

In the special case \( P_n = 1, k = 1 \) and \( \delta = 0 \) for all values of \( n, |N, p_n, \delta|_k \) summability is reduces to \( |N, p_n|_k \)-summability.

A positive sequence \((b_n)\) is said to be almost increasing if there exist a positive increasing sequence \((c_n)\) and two positive constants \(A\) and \(B\) such that

\[
Ac_n \leq b_n \leq Bc_n
\]

A positive sequence \((\gamma_n)\) is said to be quasi-\(\beta\)-power increasing sequence if there exist a constant \(k = k(\beta, \gamma) \leq 1\) such that

\[
k^\beta \gamma_n = m^\beta \gamma_m
\]

Hold for all \( n \geq m + 1 \). It should be noted the every almost increasing sequence is a quasi-\(\beta\)-power increasing sequence for any non-negative \(\beta\) and converse is not true.

II. Known Results:

BOR [2] has proved the following theorem for \(|N, p_n|_k\)-summability factors.

**Theorem 2.1:** Let \((X_n)\) be a positive non-decreasing sequence and let there be sequences \((\beta_n)\) and \((\lambda_n)\) such that

\[
| \Delta \lambda_n | \leq \beta_n
\]

\[
\beta_n \to 0 \text{ as } n \to \infty
\]

\[
\sum_{n=1}^{\infty} n | \Delta \beta_n | X_n < \infty
\]

\[
| \lambda_n | X_n = O(1)
\]

If

\[
\sum_{n=1}^{\infty} | t_n |^k \gamma_n = O(X_n) \text{ as } n \to \infty
\]

where \((t_n)\) is the \(n\)-th \((C,1)\) mean of the sequence \((na_n)\) and \((p_n)\) is the sequence such that

\[
P_n = O(np_n)
\]

\[
P_n \Delta p_n = O(p_n p_{n+1})
\]

then the series \( \sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{np_n} \) is summable \(|N, p_n|_k, k \geq 1\).

Later BOR [4] has generalized Theorem 2.1 for the \(|N, p_n, \delta|_k\)-summability factors.

**Theorem 2.2:** Let \((X_n)\) be a positive non-decreasing sequence and the sequence \((\beta_n)\) and \((\lambda_n)\) are such that the condition (2.1) – (2.7) of Theorem (2.1) are satisfied with condition (2.5) replaced by

\[
\sum_{v=1}^{n} \left( \frac{P_v}{p_v} \right)^{\delta k} | t_v |^k = O(X_n) \text{ as } n \to \infty,
\]

If
Then the series \( \sum_{n=1}^{\infty} a_n P_n \lambda_n / n p_n \) is summable \( |N, p_n, \delta|_k \), \( k \geq 1 \) and \( 0 \leq \delta < \frac{1}{k} \).

Recently BOR [5] has proved the following theorem.

**Theorem 2.3:** Let \((X_n)\) be an almost increasing sequence. If the condition (2.1)-(2.4) and (2.6)-(2.9) are satisfied then the series \( \sum_{n=1}^{\infty} a_n P_n \lambda_n / n p_n \) is summable \( |N, p_n, \delta|_k \), \( k \geq 1 \) and \( 0 \leq \delta < \frac{1}{k} \).

### III. Main Results:

The aim of this paper is to prove the Theorem 2.3 under more weaker conditions for this we use the concepts of quasi-\( \beta \)-power increasing sequence. Now we shall prove the following theorem.

**Theorem 3.1:** Let \((X_n)\) be quasi-\( \beta \)-power increasing sequence if the condition (2.1)-(2.4) and (2.6)-(2.9) are satisfied, then the series \( \sum_{n=1}^{\infty} a_n P_n \lambda_n / n p_n \) is summable \( |N, p_n, \delta|_k \), \( k \geq 1 \) and \( 0 \leq \delta < \frac{1}{k} \).

### IV. Lemma:

We need the following lemma for the proof of Theorem 3.1.

**Lemma 4.1:** (LEINDER [9]) Under the condition on \((X_n), (\lambda_n)\) and \((\beta_n)\) where \((X_n)\) is quasi-\( \beta \)-power increasing sequence as taken in the statement of the theorem the following condition holds

\[
O(1) \text{ as } n \to \infty.
\]

and

\[
\sum_{n=1}^{\infty} \beta_n X_n < \infty.
\]

**Lemma 4.2:** (BOR [2]) If condition (2.6) and (2.7) are satisfied then we have

\[
\Delta \left( \frac{P_n}{n^2 p_n} \right) = O \left( \frac{1}{n^2} \right)
\]

**Lemma 4.3:** (BOR [2]) If the condition (2.1)-(2.4) are satisfied, then we have

\[
\lambda_n = O(1)
\]

\[
\Delta \lambda_n = O \left( \frac{1}{n} \right)
\]

### 2.5 PROOF OF THE THEOREM:

Let \((T_n)\) be the sequence of \( |N, P_n| \) means of the series \( \sum_{n=1}^{\infty} a_n P_n \lambda_n / n p_n \). Then we have

\[
T_n = \frac{1}{P_n} \sum_{v=1}^{n} P_v \sum_{r=1}^{v} a_r P_r \lambda_r / r p_r
\]

\[
= \frac{1}{P_n} \sum_{v=1}^{n} (P_v - P_{v-1}) a_v P_v \lambda_v / v p_v
\]

then, for \( n \geq 1 \)
\[ T_n - T_{n-1} = \frac{p_n}{p_{n-1}} \sum_{v=1}^{n} \frac{P_{v-1}a_{v}v^{k} \lambda_{v}}{v^{2}p_{v}} \]

Using Abel’s transformation, we get

\[ T_n - T_{n-1} = \frac{p_n}{p_{n-1}} \sum_{v=1}^{n} \frac{P_{v}a_{v}v^{k} \lambda_{v}}{v^{2}p_{v}} = \frac{p_n}{p_{n-1}} \sum_{v=1}^{n} \frac{P_{v}a_{v}v^{k} \lambda_{v}}{v^{2}p_{v}} \]

To complete the proof of the theorem by Minkowski’s inequality, it is sufficient to show that

\[ \sum_{n=1}^{\infty} \left( \frac{P_{n}}{p_{n}} \right)^{\delta k + k-1} |T_{n,r}|^{k} < \infty, \quad \text{for } r = 1, 2, 3, 4 \quad (5.2) \]

Now, applying Hölder’s inequality, we have that

\[ \sum_{n=2}^{m+1} \left( \frac{P_{n}}{p_{n}} \right)^{\delta k + k-1} |T_{n,1}|^{k} = O(1) \sum_{n=2}^{m+1} \left( \frac{P_{n}}{p_{n}} \right)^{\delta k - 1} \left\{ \sum_{v=1}^{n-1} \frac{P_{v}}{p_{v}} |t_{v}| |\lambda_{v}|^{k} \right\}^{k} \]

\[ = O(1) \sum_{n=2}^{m+1} \left( \frac{P_{n}}{p_{n}} \right)^{\delta k - 1} \left\{ \sum_{v=1}^{n-1} \frac{P_{v}}{p_{v}} |t_{v}| |\lambda_{v}|^{k} \right\}^{k} \left[ \sum_{v=1}^{n-1} \frac{P_{v}}{p_{v}} \right]^{k-1} \]

\[ = O(1) \sum_{v=1}^{m} \left( \frac{P_{v}}{p_{v}} \right)^{k} |t_{v}| |\lambda_{v}|^{k} \frac{1}{v^{k}} \sum_{v=n+1}^{m+1} \left( \frac{P_{v}}{p_{v}} \right)^{\delta k - 1} \left[ \sum_{v=1}^{n-1} \frac{P_{v}}{p_{v}} \right]^{\delta k} \]

\[ = O(1) \sum_{v=1}^{m} \left( \frac{P_{v}}{p_{v}} \right)^{k} |t_{v}| |\lambda_{v}|^{k} \frac{1}{v^{k}} \left( \frac{P_{v}}{p_{v}} \right)^{\delta k} \]

\[ = O(1) \sum_{v=1}^{m} \left( \frac{P_{v}}{p_{v}} \right)^{k} |t_{v}| |\lambda_{v}|^{k} \frac{1}{v^{k}} \left( \frac{P_{v}}{p_{v}} \right)^{\delta k} \]

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\[ \sum_{v=1}^{m} \Delta \lambda_v \left( \frac{|t_r|}{r} \right)^k + O(1) \right) \]
= O(1) as \( m \to \infty \)

Next,

\[
\sum_{n=2}^{m+1} \left( \frac{P_n}{P_n} \right)^{\delta k + k - 1} |T_{n,3}|^k
\]

\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{P_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} P_v \lambda_{v+1} \left| t_v \right| \left( \frac{v+1}{v} \right)^k \right\}
\]

\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{P_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} P_v \lambda_{v+1} \left( \frac{1}{v} \right)^k \left| t_v \right| \right\}
\]

\[
= O(1) \sum_{n=2}^{m+1} \left( \frac{P_n}{P_n} \right)^{\delta k - 1} \frac{1}{P_{n-1}^k} \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} P_v \lambda_{v+1} \left| t_v \right| \right\}^{k-1}
\]

\[
= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{P_v} \right)^{k-1} \frac{1}{P_{v-1}^k} \left| \frac{P_v}{P_v} \right|^{\delta k}
\]

\[
= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{P_v} \right)^{\delta k - 1} \frac{1}{P_{v-1}^k} \left| \frac{P_v}{P_v} \right|^{\delta k}
\]

\[
= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{P_v} \right)^{\delta k} \left| t_v \right|^{k} \left( \frac{P_v}{P_v} \right)^{\delta k}
\]

\[
= O(1) \sum_{v=1}^{m} \left( \frac{P_v}{P_v} \right)^{\delta k} \left| t_v \right|^{k} \left( \frac{P_v}{P_v} \right)^{\delta k}
\]

\[
= O(1) \sum_{v=1}^{m} \beta_v X_v + O(1) \lambda_{m+1} |X_m|
\]

\[
= O(1) \sum_{v=1}^{m} \beta_v X_v + O(1) \lambda_{m+1} |X_m|
\]

\[
= O(1) \sum_{v=1}^{m} \beta_v X_v + O(1) \lambda_{m+1} |X_m|
\]

\[
= O(1) \text{ as } m \to \infty
\]

Finally

\[
\sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta k + k - 1} |T_{n,4}|^k
\]

\[
= O(1) \sum_{n=1}^{m} \left( \frac{P_n}{P_n} \right)^{\delta k} \left( \frac{P_n}{P_n} \right)^{k-1} \left( \frac{n+1}{n} \right)^k \frac{1}{n^k} |\lambda_n|^k |t_n|^k
\]
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\[ O(1) = \sum_{n=1}^{m} \left( \frac{p_n}{p_n} \right) \delta k \left| n^{k-1} \frac{1}{n^k} | \lambda_n | t_n |^k | N_n | \right. \]

\[ = O(1) \sum_{n=1}^{m} \left| \lambda_n \right| \left( \frac{p_n}{p_n} \right) \delta k \left| t_n \right|^k \]

\[ = O(1) \text{ as } m \to \infty. \]

Therefore, we get that

\[ \sum_{n=1}^{m} \left( \frac{p_n}{p_n} \right) \delta k + k-1 \left| T_{n,r} \right|^k = O(1) \text{ as } m \to \infty \text{ for } r = 1, 2, 3, 4. \]

This completes the proof of the theorem.

V. Corollary:

Our theorem has following results as a corollary.

**Corollary 6.1:** Taking \( \delta = 0 \) in theorem 4.1, we get Theorem 2.1 as a corollary. Since for \( \delta = 0, |N, p_n, \delta|_k \) summability reduces to \( |N, p_n|_k \)-summability.

VI. Conclusion:

Our theorem have the more general result rather than any previous known results. So our theorem enrich the literature of summability theory.

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References: