Topological 3-Rings

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Abstract: In this paper we study the 3-rings, Idempotent of 3-ring and some other theorems. In the second section we introduce Ideals on 3-rings, center of 3-rings and theorems, Topological 3-rings and their properties: the set of open neighbourhoods of 0, its properties in topological 3-rings, Every topological 3-ring is a homogeneous algebra and other theorem.

Key words: Hausdorff space, Ring, p-ring, Topological space.

I. Introduction

D. Van Dantzig firstly introduced the concept of topological ring in his thesis. Later N. Jacobson, L.S. Pontryagin, L.A. Skornjakov Small and S. Warner developed and studied various properties: Connected topological rings, Totally disconnected topological rings, Banach algebras, Ring of P-addict integers, locally compact fields, locally compact division rings and their structure. McCoy and Montgomery introduced the concept of a p-ring (p prime) as a ring R in which \( x^p = x \) and \( p^x = 0 \) for all \( x \) in \( R \). Thus, Boolean rings are simply 2-rings \( (p = 2) \). Koteswararao.P in his thesis developed the concept of 3-rings, 3-rings generates \( \mathbb{A}^* \)-algebras and their equivalence. With this as motivation, I introduce the concept of Topological 3-rings.

1. Preliminaries

1.1 Definition: A commutative ring \((R, +, \cdot)\) such that \( x^3 = x, 3x = 0 \) for all \( x \) in \( R \) is called a 3-ring.

1.2 Note: (1) \( x + x = -x \) for all \( x \) in a 3-ring \( R \)

(2) Here after \( R \)-stands for a 3-ring.

1.3 Example: \( 3 = \{0, 1, 2\} \). Then \((3, +, \cdot, 0, 1)\) is a 3-ring where

\[
\begin{array}{ccc}
+ & 0 & 1 & 2 \\
0 & 0 & 1 & 2 \\
1 & 1 & 2 & 0 \\
2 & 2 & 0 & 1
\end{array}
\]

1.4 Example 2: Suppose \( X \) is a non empty set. Then \((3^X, +, \cdot, 0.1)\) is a 3-ring with

(a) \((f + g)(x) = f(x) + g(x)\).
(b) \((f \cdot g)(x) = f(x) \cdot g(x)\).
(c) \(0(x) = 0\).
(d) \(1(x) = 1\) for all \( x \in X \), \( f, g \in 3^X \).

1.5 Definition: Let \( R \) be a 3-ring. An element \( a \in R \) is called an idempotent if \( a^2 = a \).

1.6 Lemma: An element \( a \in R \) is an idempotent iff \( 1-a \) is an idempotent.

Proof: Suppose \( a \) is an idempotent

Claim : \((1-a)\) is an idempotent

\[
(1 - a)^2 = 1 + a^2 - 2a = 1 + a - 2a \quad (\because \text{a is idempotent})
\]

\[
= 1 - a
\]

\(\therefore (1-a) \) is an idempotent

Conversely suppose that \((1-a)\) is an idempotent

We have to show that \( a \) is an idempotent:

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1.7 Lemma: For any element \( a \) in a 3-ring \( R \), \( a^2 \) is an idempotent. 

**Proof:** Suppose \( a \in R \).
\[ \because R \text{ is a 3-ring}, \ a^3 = a \]
\[ (a^2)^2 = a^2, \ a^2 = a^3, \ a = a, \ a = a^2. \]
\[ \because a^2 \text{ is an idempotent for every } a \in R. \]

### II. Main Results

2.1 Definition: A non empty subset \( I \) of a 3-ring \( R \) is said to be ideal if

1. \( a, b \in I \Rightarrow a + b \in I \),
2. \( r \in R \Rightarrow ar, ra \in I \).

**Note:** A non empty subset \( I \) of \( R \) is said to be a right ideal (left ideal) of \( R \), if

1. \( a, b \in I \Rightarrow a + b \in I \),
2. \( a \in I, r \in R \Rightarrow ar \in I \) (\( r a \in I \)).

2.1 Note: Suppose \( a \in R \) then there is a minimal right ideal (left ideal) exists containing \( a \) which is called the principal right (left) ideal denoted by \( (a) \).

### 2.2 Note:
Suppose \( a \in R \) then there is a minimal right ideal (left ideal) exists containing \( a \) which is called the principal right (left) ideal denoted by \( (a) \).

\[ (a)_r = \{ar / r \in R \} \text{ and } (a)_l = \{ra / r \in R \}. \]

2.2 Note: For any set of ideals \( I_1, I_2, \ldots. \) \( \exists \) a maximal ideal \( I \) such that \( I \subseteq I_1, I_2, \ldots \) and it is denoted by \( \text{glb} \{I_1, I_2, \ldots \}. \)

2.2 Note: For any set of ideals \( I_1, I_2, \ldots. \) \( \exists \) a minimal ideal \( I \) such that \( I \subseteq I_1, I_2, \ldots \) and it is denoted by \( \text{lub} \{I_1, I_2, \ldots \}. \)

2.2 Note: For the ideals \( I_1, I_2 \) \( \text{glb} \{I_1, I_2\} \) is denoted by \( I_1 \wedge I_2 \) and \( \text{lub} \{I_1, I_2\} \) is denoted by \( I_1 \vee I_2 \).

Thus the set of right ideals form a lattice with \( \wedge, \vee \) Zero (0), unit \( R \).

2.4 Definition: The center of a 3-ring \( R \) is the set \( C = \{a \in R / ax = xa, \forall x \in R \} \). C is a commutative ring with unit 1.

2.7 Theorem: If \( a, b \) are the idempotent elements in \( C \), then \( ab \) is an idempotent and \( ab \in C \) and also \( (a) \wedge (b) = (ab) \).

**Proof:** Let \( R \) be a 3-ring.

Suppose \( a, b \in R \) and \( a, b \) are idempotents.
\[ (ab)^2 = ab \cdot ab = a^2 \cdot b^2 = ab. \]
Therefore \( ab \) is an idempotent.

Let \( x \in R \Rightarrow (ab)x = a(bx) = (ax)b = (xb)a = x(ab) \)
Therefore \( (ab)x = x(ab) \)
\[ \therefore ab \in C. \ ab = ba \in a \text{ and belongs to } b \]
\[ \therefore (ab) \subseteq (a) \wedge (b). \]

2.8 Theorem: If \( a, b \) are idempotents in \( C \), then \( a+b-ab \in C \), idempotent and also \( (a) \vee (b) = (a+b-ab). \)

**Proof:** \( a+b-ab = 1-(1-a)(1-b). \)
Since \( a, b \in C \Rightarrow (1-a)(1-b) \in C \) and are idempotent.
\[ \Rightarrow 1-(1-a)(1-b) \in C \text{ and idempotent.} \]
\[ \therefore a + b-a b \text{ is an idempotent and belongs to } C. \]
(a) \cdot \lor (b) = ((a) \cdot \lor (b)) \cdot = (1 - (1 - a)(1 - b)) \cdot = (a - b - ab).
\therefore (a) \cdot \lor (b) = (a + b - ab).

2.9 Theorem: Center of a 3-ring C is a 3-ring

Proof: Let a \in C \Rightarrow a \in R
Since R is a 3-ring and a \in R then a^3 = a and 3a = 0
We have
3(a x) = (3a)x
= 0x \quad \text{(since R is a 3-ring)}
= 0.
Therefore 3(a x) = 0, \forall a \in R, x \in R.
Let z \in R
z \cdot a x = z a x
= x z a
= x a z
= a x z.
\therefore a x \in C.
Therefore C is 3-ring.

2.10 Definition: A set R is said to be a topological 3-ring if
1. R is a 3-ring.
2. R is a topological space.
3. The operations +, ., \cdot, \cdot* are continuous.

2.11 Note: For any subsets U, V \subseteq R, we define
U + V = \{u + v/u \in U, v \in V\}
U \cdot V = \{uv/u \in U, v \in V\}.
– U = \{-u/u \in U\}.
U* = \{u*/u \in U\}.

2.12 Note: 1) + : R \times R \to R is continuous means, for every neighbourhood W of a + b, a, b \in R there exist
neighbourhoods U of a, V of b such that U + V \subseteq W.

2) \cdot : R \times R \to R is continuous means, for every neighbourhood W of ab, a, b \in R, there exist
neighbourhoods U of a, V of b such that
U \cdot V \subseteq W.

3) – : R \to R is continuous, if for every neighbourhood W of –a, there exist a neighbourhood U of
a such that – U \subseteq W.

4) (\cdot*) : R \to R is continuous means, for every neighbourhood W of a*, there exist a
neighbourhood U of a such that U* \subseteq W.

2.13 Lemma: Suppose R is a topological 3-ring. If c \in R, then
i) The map x \to x c + x, is homeomorphism.
ii) The maps x \to x c, x \to x c x are continuous.

Proof: The subspace \{c\} \times R of R \times R is clearly homeomorphic to R via (c, b) \to c + b and the restriction of +
to \{c\} \times R to R is continuous and clearly bijection.
4.8 \[\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}, (c, x) \mapsto c + x\] is continuous and bijective.

The subspace \(\{c\} \times \mathbb{R}\) of \(\mathbb{R} \times \mathbb{R}\) is clearly homeomorphic to \(\mathbb{R}\) via \((c, b) \mapsto b\) and the restriction of \(\mathbb{R}\) to \(\{c\} \times \mathbb{R}\) is continuous. Similarly \(x \mapsto xc\) is continuous.

2.14 Note : 1) \(\mathbb{R}\) is a topological 3- ring. Since \(-: \mathbb{R} \rightarrow \mathbb{R}\) is clearly homeomorphism. So \(U\) is open, \(-U\) is also open.

2) Since \(\mathbb{R} \rightarrow \mathbb{R} + c\) is homeomorphic, then for any open \(U \subseteq \mathbb{R}\), \(c \in \mathbb{R}\), then \(U + c = \{u + c/u \in U\}\) is open. If \(U, V\) are open, then \(U + V\) is open.

3) If \(U\) is open neighbourhood of \(c\) if \(-U - c\) is an open neighbourhood of \(0\). So, the topology of \(\mathbb{R}\) is completely determined by the open neighbourhoods of \(0\).

2.15 Definition : Let \(X\) be a topological space. If \(x \in \mathbb{R} \times X\), then a fundamental system of neighbourhoods of \(x\) is a non-empty set \(N\) of open neighbourhoods of \(x\) with the property that, if \(U\) is open and \(x \in N \subseteq U\), then there is \(V \in N\) such that \(V \subseteq U\).

2.16 Definition : Let \(R\) be a 3- ring. A non-empty set \(N\) of subsets of \(R\) is fundamental if it satisfies the following conditions.

(a) Every element of \(N\) contains 0.
(b) If \(U, V \subseteq N\), then there is \(W \subseteq N\) with \(W \subseteq U \cap V\).
(c) If \(U \subseteq N\) and \(c \in U\), then there exist \(V \in N\) such that \(c + V \subseteq U\).
(d) If \(c \in N\) then \(c + U = U\).
(e) If \(U \subseteq N\), then \(-U \subseteq N\).
(f) If \(U \subseteq N\) there exist \(V \in N\) such that \(V \subseteq U\).
(g) For \(c \in R\) and \(U \subseteq N\) there is \(V \in N\) such that \(c \in V \subseteq U\).
(h) For each \(U \subseteq N\) there is \(V \in N\) such that \(V \subseteq U\).

2.17 Theorem : Suppose \(R\) is a topological 3- ring. Then the set \(N\) of open neighbourhoods of \(0\) satisfies.

(a) Every element of \(N\) contains 0 and \(0\) is an open set.
(b) \(0 \in N\) and \(c \in U\), then \(0 \in V \subseteq N\) such that \(c + V \subseteq U\).
(c) If \(U \subseteq N\), then \(-U \subseteq N\).
(d) \(U \subseteq N\) there exist \(V \in N\) such that \(V \subseteq U\).
(e) For each \(U \subseteq N\) there is \(V \subseteq N\) such that \(V \subseteq U\).

Conversely, if \(R\) is a regular ring and \(N\) a non-empty set of subsets of \(R\) which satisfies \(N0, N1, N2, N3, N4, N5\) has the property that (a) every element of \(N\) contains 0 and (b) if \(U, V \subseteq N\), then there is \(W \subseteq N\) such that \(W \subseteq U \cap V\), then there is a unique topology on \(R\) making \(R\) into a topological 3-ring in such away that \(N\) is a fundamental system of neighbourhoods of \(0\).

Proof :

\(N0\) . Let \(U \subseteq N\) and \(c \in U\), then \(-U\) is a neighbourhood of \(0\).

\(N1\). Let \(U \subseteq N\) and \(0 \in U\), then \((-U) = (0, 0) \in U + V \subseteq U\) and \(V \subseteq U\).

\(N2\). Let \(U \subseteq N\) and \(0 \in U\) neighbourhood of \(0\).

\(N3\). If \(U \subseteq N\) and \(0 \in U\) is a neighbourhood of \(0\), then the ring \(R\) is homeomorphic and \(U\) is open, then \(U\) is open.

\(N4\). If \(0 \in U\) and \(U \subseteq N\), then \(V \subseteq N\) such that \(V \subseteq U\).

\(N5\). If \(0 \in U\) and \(U \subseteq N\), then \(V \subseteq N\) such that \(V \subseteq U\).

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Let $U \subseteq \mathbb{N}$ and $c \in \mathbb{R} \Rightarrow \forall \alpha \in \mathcal{X}$ is continuous and $U$ is a subset of $\mathbb{R}$, then $\forall \alpha \in \mathcal{X}$ is continuous.

Let $U \subseteq \mathbb{N}$ and $c \in \mathbb{R} \Rightarrow \forall \alpha \in \mathcal{X}$ is continuous and $U$ is a subset of $\mathbb{R}$, then $\forall \alpha \in \mathcal{X}$ is continuous.

For $\alpha \subseteq \mathcal{X}$, $\mathcal{X}$ is open set.

Claim : If $U$ is open, then for $c \in \mathbb{C}$, $c + U$ is open.

Let $b \subseteq \mathbb{C} + U \Rightarrow \forall \alpha \in \mathcal{X}$ is continuous. $\forall \alpha \in \mathcal{X}$ is continuous.

Claim : $+: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous.

Let $(c, d) \subseteq \mathbb{R} \times \mathbb{R} \Rightarrow \forall \alpha \in \mathcal{X}$ is continuous. $\forall \alpha \in \mathcal{X}$ is continuous.

Claim : $\times : \mathbb{R} \rightarrow \mathbb{R}$ by $x \mapsto cx$ is continuous.

Let $U \subseteq \mathbb{N}$ and $c \in \mathbb{R} \Rightarrow \forall \alpha \in \mathcal{X}$ is continuous and $U$ is a subset of $\mathbb{R}$, then $\forall \alpha \in \mathcal{X}$ is continuous.

Claim : $\mathbb{R} \subseteq \mathbb{R}$ and $\forall \alpha \in \mathcal{X}$ is continuous.

Let $U \subseteq \mathbb{N}$ and $c \in \mathbb{R} \Rightarrow \forall \alpha \in \mathcal{X}$ is continuous and $U$ is a subset of $\mathbb{R}$, then $\forall \alpha \in \mathcal{X}$ is continuous.

Claim : $m : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, where $m(a, b) = a \cdot b$

Let $U$ be an open set and $(c, d) \subseteq \mathbb{R} \times \mathbb{R}$.

The maps $\theta : \mathbb{R} \rightarrow \mathbb{R}$ and $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ are continuous.

Claim : $\mathbb{R} \subseteq \mathbb{R}$ and $\forall \alpha \in \mathcal{X}$ is continuous.

Let $U \subseteq \mathbb{N}$ and $c \in \mathbb{R} \Rightarrow \forall \alpha \in \mathcal{X}$ is continuous and $U$ is a subset of $\mathbb{R}$, then $\forall \alpha \in \mathcal{X}$ is continuous.

Claim : $\mathbb{R} \subseteq \mathbb{R}$ and $\forall \alpha \in \mathcal{X}$ is continuous.
Let \( P_d = \Psi^{-1}(P) \cap \overline{Q} \cap V \), \( P_c = \Theta^{-1}(P) \cap \overline{Q} \cap V \).

Then \((c, d) \in (c + P_d) \times (d + P_c) \subseteq \bigcap m \downarrow (U). \therefore m \) is continuous.

**Claim:** \( \star: R \rightarrow \overline{R} \) is continuous. Let \( U \) be an open set.

Let \( x \in \Psi^{-1}(U) \Rightarrow x \in \overline{U} \Rightarrow x \) is a neighbourhood of \( 0 \).

\( \Rightarrow \exists x \in \overline{V} \in N \exists V \subseteq \overline{U} - x \Rightarrow x + V \subseteq \overline{U} \).

Suppose \( \overline{N} \) is another topology on \( R \) for which \( N \) is a fundamental system of neighbourhoods of \( 0 \) in this topology. Then the topology \( \overline{\overline{N}} \) and the topology defined above have same open base.

\( \therefore \) The topology \( \overline{\overline{N}} \) must agree with the topology we have defined above.

\( \therefore \) The topology is unique.

\( \therefore \overline{N} \) generates a unique topology on \( R \) for which \( N \) is a fundamental system of neighbourhoods of \( 0 \).

2.18 **Note:** If \( R \) is a 3- ring then \( R \) has no non-zero nilpotent elements, every prime ideal is maximal and Jacobson radical of \( R \) is \( \{0\} \).

2.19 **Theorem:** Suppose \( R \) is a topological 3- ring. \( S, T \) are subsets of \( R \).

Then
a) \( ST, S + T \) are compact whenever \( S, T \) are compact.

b) \( -S, S^* \) are compact whenever \( S \) is compact.

c) \( ST, S + T \) are connected sets whenever \( S, T \) are connected sets.

d) \( -S, S^* \) are connected whenever \( S \) is connected.

**Proof:**

a) Since continuous image of a compact set is compact.

\( +: R \times R \rightarrow R \) are continuous, \( S, T \) are compact sets, then

\( (S \times T) = ST, \)

\( +(S \times T) = S + T \) are compact.

b) \( -: R \rightarrow \overline{R} \) and \( *: R \rightarrow \overline{R} \) are continuous and \( S \) is compact, then \( -S, S^* \) are compact.

c) \( \overline{-} \) continuous image of a connected set in connected,

\( .: R \times R \rightarrow \overline{R} \) and \( +: R \times R \rightarrow \overline{R} \) are continuous,

\( (S \times T) = ST, +(S \times T) = S + T \) are connected sets.

d) \( \overline{-}: R \rightarrow \overline{R} \) and \( \star: R \rightarrow \overline{R} \) are continuous and \( S \) is connected, so \( -S, S^* \) are connected.

2.20 **Theorem:** The union of all connected subsets contain 0 is a topological 3- ring.

**Proof:** Suppose \( \{S_i \mid i \in I\} \) is a class of all connected sets containing 0.

Let \( S = \bigcup_i S_i \) contain 0

\( i \in I \).

\( \therefore \) 0 \( \subseteq S \Rightarrow \exists 0 \in S_i \) for some \( i \in I \Rightarrow 0 \in S_i \Rightarrow \exists 1 \in \overline{S_i} \)

\( \forall S_i \) is connected, \(-S_i \) is also connected.

\( \forall S_i \subseteq S \Rightarrow \forall a \in K \) for some \( i \Rightarrow \exists a \in -K \) \( i \Rightarrow a \in S_i \).

\( \forall \exists a \in S \Rightarrow \forall a \in S \Rightarrow \exists b \in S_j \Rightarrow a + b \in S_i + S_j \)

\( \forall S \subseteq S_i (\forall S_i + S_j \) is connected)

\( \forall S \) is a topological sub 3- ring of \( R \).

2.21 **Theorem:** Suppose \( R \) is a topological 3- ring and \( I \) is ideal of \( R \). Then

\( \overline{I} \) is also ideal of \( R \).

**Proof:** Suppose \( I \) is an ideal of \( R \). \( \overline{I} = \{a \in \overline{R}/\forall \) every neighbourhood of \( a \) intersects \( I\} \)

**Claim:** \( \overline{I} \) is an ideal. Let \( a, b \in \overline{I} \)

\( \Rightarrow \overline{I} \) every neighbourhood of \( a \), every neighbourhood of \( b \) intersects \( I \).

Suppose \( W \) is a neighbourhood of \( a + b \).

\( \Rightarrow \exists U \) neighbourhood of \( a \), neighbourhood \( V \) of \( b \) such that \( U + V \subseteq \overline{W} \).

\( \forall U \) intersects \( I \), \( V \) intersects \( I \) so \( U + V \) intersects \( I \), then \( W \) intersects \( I \).

\( \forall a + b \in \overline{I} \). Let \( a \in \overline{I} \), \( b \in R \).

**Claim:** \( a, b \in \overline{I} \Rightarrow \overline{a} \in \overline{I} \) every neighbourhood of \( a \) intersects \( I \).

Let \( W \) be a neighbourhood of \( ab \). then \( \exists U \) neighbourhood of \( a \), neighbourhood \( V \) of \( b \exists U \subseteq \overline{W} \).

\( \forall U \cap \overline{I} \neq \overline{I} \). \( \forall a \in \overline{I} \). \( \exists a \in \overline{I} \) \( \exists a \in \overline{I} \) \( \forall U \subset \overline{I} \)

\( \exists UV \cap \overline{I} \neq \overline{I} \) \( \exists UV \cap \overline{I} \neq \overline{I} \)

\( \forall UV \subseteq W \), \( \forall W \) intersects \( I \), \( \forall a \in \overline{I} \)

Similarly \( ba \in \overline{I} \). \( \forall a \in \overline{I} \) is an ideal of \( R \).
2.22 Theorem: Every maximal ideal M of a topological 3-ring R is closed.

Proof: Clearly M ⊆ M, so M = M is ideal, so M = M is closed.

2.23 Theorem: If a topological 3-ring is T2 space then it is a Hausdorff space.

Proof: Suppose R is a T2 space and a, b ∈ R and a ≠ b.

→ R is a T2 space 3-neighbourhood U of a and neighbourhood V of b ∋ a ∉ V, b ∉ U. Suppose U ∩ V ≠ ∅.

Let W = U ∩ V. Let c ∈ W = W is a neighbourhood of 0.

Let K = W − c = K is a neighbourhood of 0.

→ K + a and K + b are neighbourhoods of a and b respectively and

(K + a) ∩ (K + b) = ∅.

⇒ R is a Hausdorff space.

2.24 Theorem: Every topology 3-ring is a homogeneous algebra.

ie., for every p, q (p ≠ q) there is a continuous map f : R → R such that f(p) = q.

Proof: R is a topology 3-ring. Let c = q − p, then the function f : R → R by f(x) = c + x is continuous and f(p) = c + p = q − p + p = q.

2.25 Theorem: Suppose R is a topology 3-ring and X = Spec R. R* is a complete Boolean algebra. Suppose M is a subset of Spec R = X.

Denote QM the set of elements e ∈ R* for which M ⊆ X e. Then X ∩ QM ⊆ M. In particular if M is nowhere dense in Spec R, then ∩ QM = 0.

Proof: Let x ∈ X ∩ QM.

Suppose x ∈ M = X e (e' = 1 − e) = X QM

→ c ∩ (c ∩ QM) = e = 0 i.e., c ∩ (c ∩ QM) = 0

→ e ∩ (c ∩ QM) = 0

It is a contradiction (Q x ∈ X ∩ QM and x ∈ X e).

⇒ x ∈ M contains no non-empty open subset.

But X ∩ QM ⊆ M = X ∩ QM = 0.

Reference