Measure space on Weak Structure

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Abstract: Császár in [4] introduce a weak structure as generalization of general topology. The aim of this paper is to give basic concepts of the measure theory in weak structure.

Keywords: weak structure, σ-algebra, σ-additive function, Measures

I. Notation and Preliminaries

In mathematical analysis. Measurement theory plays a vital role in the expression completely for some mathematical concepts. In our research, we introduced some of the concepts of measurement in a weak structure. And we study their properties and some applications it. So we shall denote by \( X \) a nonempty set, by \( \omega \) a weak structure [1] and by \( P(X) \) the set of all parts (i.e., subsets) of \( X \), and by \( \emptyset \) the empty set. For any subset \( \lambda \) of \( X \) we shall denote by \( \lambda^c \) its complements, i.e., \( \lambda^c = \{ x \in X | x \notin \lambda \} \). For any \( \lambda, \mu \in P(X) \) we set \( \lambda|\mu = \lambda \cap \mu^c \). Let \( (\lambda_n) \) be a sequence in \( P(X) \).

The following Demorgan identity holds \( (\bigcup_{n=1}^{\infty} \lambda_n) = \bigcap_{n=1}^{\infty} \lambda_n^c \). We define

\[
\lim_{n \to \infty} \bigvee \lambda_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \lambda_m, \quad \lim_{n \to \infty} \bigwedge \lambda_n = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \lambda_m.
\]

If \( L = \lim_{n \to \infty} \bigvee \lambda_n = \lim_{n \to \infty} \bigwedge \lambda_n \), then we set \( L = \lim_{n \to \infty} (\lambda_n) \), and we say that \( (\lambda_n) \) converges to \( L \).

As easily checked, \( \lim_{n \to \infty} \bigvee \lambda_n \) (resp., \( \lim_{n \to \infty} \bigwedge \lambda_n \)) consists of those elements of \( X \) that belong to infinite elements of \( (\lambda_n) \) (resp., that belong to infinite elements of \( (\lambda_n) \)) expect perhaps a finite number. Therefore, \( \lim_{n \to \infty} \bigwedge \lambda_n \subset \lim_{n \to \infty} \bigvee \lambda_n \).

And it easy also to check that, if \( (\lambda_n) \) is increasing \( (\lambda_n \subset \lambda_{n+1}, n \in N) \), then \( \lim_{n \to \infty} \lambda_n = \bigcup_{n=1}^{\infty} \lambda_n \) where, if \( (\lambda_n) \) is decreasing \( (\lambda_n \supset \lambda_{n+1}, n \in N) \), then \( \lim_{n \to \infty} \lambda_n = \bigcap_{n=1}^{\infty} \lambda_n \).

In the first case we shall write \( \lambda_n \uparrow L \), and in the second \( \lambda_n \downarrow L \).

II. Algebra and \( \sigma \)-algebra on a weak structure \( \omega \)

Let \( A \) be a nonempty subset of \( \omega \)

Definition 1.1 \( A \) is said to be an algebra in \( \omega \) if

a) \( \phi \in A \)
b) \( \lambda, \mu \in A \Rightarrow \lambda \cup \mu \in A \)
c) \( \lambda \in A \Rightarrow \lambda^c \in A \)

Remark 1.1 It easy to see that, if \( A \) is an algebra and \( \lambda, \mu \in A \), then \( \lambda \cup \mu \) and \( \lambda|\mu \) belong to \( A \). Therfore, the symmetric difference \( \lambda \Delta \mu = (\lambda|\mu) \cup (\mu|\lambda) \) also belong to \( A \). Moreover, \( A \) is stable under finite union and intersection,

that is \( \lambda_1, \ldots, \lambda_n \in A \Rightarrow \lambda_1 \cup \cdots \cup \lambda_n \in A \)

\( \lambda_1 \cap \cdots \cap \lambda_n \in A \).
Definition 1.2 An algebra $A$ in $\omega$ is said to be a $\sigma$-algebra if, for any sequence $(\lambda_n)$ of elements of $A$, we have $\bigcup_{n=1}^{\infty} \lambda_n \in A$. We note that, if $A$ is $\sigma$-algebra and $(\lambda_n) \subseteq A$, then $\bigcap_{n=1}^{\infty} \lambda_n \in A$ owing to the De Morgan identity. Moreover, $\lim_{n \to \infty} (\bigwedge \lambda_n) \in A$, $\lim_{n \to \infty} (\bigvee \lambda_n) \in A$.

The following examples explain the difference between algebras and $\sigma$-algebras.

Example 1.1 Obviously, $P(X)$ and $\mathcal{E} = \{\phi\}$ are $\sigma$-algebras in $X$. Moreover, $\omega$ is the largest $\sigma$-algebras in $X$, and $\mathcal{E}$ is the smallest.

Example 1.2 In $[0,1)$, the class $\rho$ consisting of $\phi$ and of all finite unions $\beta = \bigcup_{i=1}^{n} [a_i, b_i)$ with $0 \leq a_i \leq b_i \leq a_{i+1} \leq 1$ is an algebra.

Example 1.3 In an infinite set $X$ consider the class $\rho = \{\theta \in \omega | \theta$ is finite, or $\theta^c$ is finite $\}$. Then $\rho$ is an algebra.

Example 1.4 In an uncountable set $X$ consider the class $\rho = \{\theta \in \omega | \theta$ is uncountable, or $\theta^c$ is uncountable $\}$. Then $\rho$ is a $\sigma$-algebra.

Definition 1.3 The intersection of all $\sigma$-algebras including $\tau \subseteq \omega$ is called the $\sigma$-algebra generated by $\tau$, and will be denoted by $\sigma(\tau)$.

Example 1.5 Let $E$ be a metric space. The $\sigma$-algebra generated by all open subsets of $E$ is called the Borel $\sigma$-algebra of $E$, and denoted by $B(E)$.

2. Measure

2.1 Additive and $\sigma$-additive functions

Let $A \subseteq \omega$ be an algebra.

Definition 2.1 Let $F : A \to [0, +\infty]$ be such that $\mu(\phi) = 0$.

(1) We say that $F$ is additive if, for any family $A_1, \ldots, A_n \in A$ of mutually disjoint sets, we have $F\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} F(A_k)$.

(2) We say that $F$ is $\sigma$-additive if, for any sequence $(A_n) \in A$ of mutually disjoint sets such that $\bigcup_{k=1}^{\infty} A_k \in A$, we have $F\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} F(A_k)$.

Remark 2.1 Let $A \subseteq \omega$ be an algebra.

(1) Any $\sigma$-additive function on $A$ is also additive.

(2) If $F$ is additive, $\lambda, \mu \in A$, and $\lambda \supseteq \mu$, then $F(\lambda) = F(\mu) + F(\lambda \setminus \mu)$.

Therefore, $F(\lambda) \geq F(\mu)$. 
(3) Let \( F \) is additive on \( A \), and let \((A_n) \in A\) be mutually disjoint sets such that \( \bigcup_{k=1}^{\infty} A_k \in A \). Then, \( F( \bigcup_{k=1}^{\infty} A_k ) \geq \sum_{k=1}^{\infty} F(A_k) \) for all \( n \in \mathbb{N} \).

Therefore, \( F( \bigcup_{k=1}^{\infty} A_k ) \geq \sum_{k=1}^{\infty} F(A_k) \)

(4) Any \( \sigma \)– additive function \( F \) on \( A \) is also countably subadditive, that is, for any sequence \((A_n) \subset A\) such that \( \bigcup_{n=1}^{\infty} A_n = X \), and \( F(A_n) < \infty \) for all \( n \in \mathbb{N} \).

(5) In view of parts 3 and 4 an additive function is \( \sigma \)– additive if and only if it is countably subadditive.

**Definition 2.2** A \( \sigma \)– additive function \( F \) on an algebra \( A \subset \omega \) is said to be

(1) finite if \( F(X) < \infty \),

(2) \( \sigma \)– finite if there exists a sequence sequence \((A_n) \subset A\) such that \( \bigcup_{n=1}^{\infty} A_n = X \), and \( F(A_n) < \infty \) for all \( n \in \mathbb{N} \).

**Example 2.1** In \( X = N \), consider the algebra \( A = \{ A \in \omega | \text{is finite, or } A^c \text{ finite} \} \). The function \( F : A \rightarrow [0, \infty] \) defined as \( F(A) = \begin{cases} n(A) & \text{if } A \text{ finite} \\ \infty & \text{if } A^c \text{ finite} \end{cases} \) (where \( n(A) \) stands for the number of elements of \( A \) is \( \sigma \)– additive. On the other hand.

The function \( F : A \rightarrow [0, \infty] \) defined as \( F(A) = \begin{cases} \sum_{n \in A} \frac{1}{2^n} & \text{if } A \text{ finite} \\ \infty & \text{if } A^c \text{ finite} \end{cases} \) is additive but not \( \sigma \)– additive.

**Theorem 2.1** Let \( \mu \) be additive on \( A \). Then \( (i) \Leftrightarrow (ii) \) where:

(i) \( \mu \) is \( \sigma \)– additive,

(ii) \((A_n)\) and \( A \subset A, A_n \uparrow A \Rightarrow \mu(A_n) \uparrow \mu(A) \).

Proof \((i) \Rightarrow (ii)\) Let \((A_n)\), \( A \subset A \), \( A_n \uparrow A \). Then, \( A = A_0 \cup \bigcup_{n=1}^{\infty} (A_{n+1} \setminus A_n) \), the above being disjoint union. Since \( \mu \) is \( \sigma \)– additive, we deduce that \( \mu(A) = \mu(A_0) + \sum_{n=1}^{\infty} (\mu(A_{n+1}) - \mu(A_n)) = \lim_{n \rightarrow \infty} \mu(A_n) \), and (ii) follows.

\[(ii) \Rightarrow (i)\] Let \((A_n) \subset A\) be a sequence of mutually disjoint sets such that \( A = \bigcup_{k=1}^{\infty} A_k \in A \). Define \( B_n = \bigcup_{k=1}^{\infty} A_k \). Then \( B_n \uparrow A \). So, in view of \((ii)\), \( \mu(B_n) = \sum_{n=1}^{\infty} \mu(A_n) \uparrow \mu(A_n) \).

This implies \((i)\)
Definition 2.2 let $\varepsilon = \{\emptyset\}$ are $\sigma$ - algebras in $X$.

1. We say that the pair $(X, \varepsilon)$ is a measurable space.
2. A $\sigma$ - additive function $\mu : \varepsilon \to [0, +\infty]$ is called a measure on $(X, \varepsilon)$
3. The triple $(X, \varepsilon, \mu)$, where $\mu$ is a measure on a measurable space $(X, \varepsilon)$ is called a measurable space
4. A measure $\mu$ is said to be complete if $A \in \varepsilon, B \subset A, \mu(A) = 0 \Rightarrow B \in \varepsilon$ (and so $\mu(B) = 0$).
5. A measure $\mu$ is said to be concentrated on a set $A \in \varepsilon$ if $\mu(A^c) = 0$.

In this case we say that $A$ is a support of $\mu$

Example 2.2 Let $X$ be a nonempty set and $x \in X$. Define for every $A \in P(X)$
$$
\delta_x(A) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \notin A 
\end{cases}
$$
Then $\delta_x$ is a measure in $X$.

References