On Generalized Projective $\phi$-Recurrent Sasakian Manifold

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Abstract: The object of the present paper is to study generalized projective $\phi$-recurrent Sasakian manifolds. Here we find a relation between the associated 1-forms A and B. We also proved that the characteristic vector field $\xi$ and vector field $\rho$ associated to the 1-forms A and B are co-directional. Finally we proved that generalized projective $\phi$-recurrent Sasakian manifold is of constant curvature.

Key Words: Generalized projective $\phi$-recurrent, Sasakian manifold, Sectional curvature.

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I. Introduction

In 1977, T. Takahashi [13] introduced the notion of locally $\phi$-symmetric Sasakian manifold and obtain few of its interesting properties. The authors like [7] and [15] have extended this notion to 3-dimensional Kenmotsu and trans-Sasakian manifolds respectively. Also $\phi$-recurrent Sasakian and Kenmotsu manifolds was studied by authors [6]. In this paper, we study generalized projective $\phi$-recurrent Sasakian manifold.

The paper is organized as follows. In preliminaries, we give a brief account of Sasakian manifolds. In section 3, we find a relation between the associated 1-forms A and B. We also proved that the characteristic vector field $\xi$ and vector field $\rho$ associated to the 1-forms A and B are co-directional. Finally we proved that a generalized projective $\phi$-recurrent Sasakian manifold is of constant curvature.

II. Preliminaries

Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a Sasakian manifold with the structure $(\phi, \xi, \eta, g)$. Then the following relations hold [1]:

(2.1) $\phi^2(X) = -X + \eta(X)\xi$, $\phi\xi = 0$,
(2.2) $\eta(\xi) = 1$, $g(X, \xi) = \eta(X)$, $\eta(\phi X) = 0$,
(2.3) $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$,
(2.4) $R(\xi, X)Y = (\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X$,
(2.5) $\nabla_\xi X = -\phi X$, $\nabla_\xi \eta(X, Y) = g(X, \phi Y)$,
(2.6) $R(X, Y)\xi = \eta(Y)X - \eta(X)Y$,
(2.7) $R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi$,
(2.8) $\eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y)$,
(2.9) $S(X, \xi) = 2\eta\eta(X)$,
(2.10) $S(\phi X, \phi Y) = S(X, Y) - 2\eta\eta(X)\eta(Y)$,

for all vector fields $X, Y, Z$ where $\nabla$ denotes the operator of covariant differentiation with respect to $g$. $\phi$ is a (1,1) tensor field, $S$ is the Ricci tensor of type $(0,2)$ and $R$ is the Riemannian curvature tensor of the manifold.

Definition 2.1. A Sasakian manifold is said to be locally $\phi$-symmetric if

$\phi^2(\nabla_w R)(X, Y)Z = 0$,

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$.

Definition 2.2. A Sasakian manifold is said to be locally projective $\phi$-symmetric if

$\phi^2(\nabla_w P)(X, Y)Z = 0$,

for all vector fields $X, Y, Z, W$ orthogonal to $\xi$.

Definition 2.3. A Sasakian manifold is said to be projective $\phi$-recurrent manifold if there exists a non-zero 1-form $A$ such that

$\phi^2(\nabla_w P)(X, Y)Z = A(W)P(X, Y)Z$,

for arbitrary vector fields $X, Y, Z, W$, where $P$ is a projective curvature tensor given by

$P(X, Y)Z = R(X, Y)Z - \frac{1}{2\eta}[S(Y, Z)X - S(X, Z)Y]$.

If the 1-form $A$ vanishes, then the manifold reduces to locally projective $\phi$-symmetric manifold.

III. Generalized Projective $\phi$-Recurrent Sasakian Manifold

Definition 3.1. A Sasakian manifold $M^{2n+1}$ is called generalized projective $\phi$-recurrent if its curvature tensor $R$ satisfies the condition

$\phi^2((\nabla_w P)(X, Y)Z = A(W)P(X, Y)Z$,
(3.1) \[ \phi^2((\nabla WP)(X,Y)Z) = A(W)P(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y], \]
where \( A \) and \( B \) are 1-forms, \( \beta \) is non-zero and these are defined by
\[ A(W) = g(W, \rho_1), B(W) = g(W, \rho_2). \]
and where \( \rho_1 \) and \( \rho_2 \) are vector fields associated with 1-forms \( A \) and \( B \) respectively.

Let us consider generalized projective \( \phi \)-recurrent Sasakian manifold. Then by virtue of
(2.1) and (3.1) we have
\[ -(\nabla WP)(X,Y)Z + \eta((\nabla WP)(X,Y)Z)\xi = A(W)P(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y]. \]

From which it follows that
\[ -g((\nabla WP)(X,Y)Z, U) + \eta((\nabla WP)(X,Y)Z)\eta(U) = A(W)g(P(X,Y)Z, U) + B(W)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \]

Let \( \{e_i\}, i = 1, 2, \ldots, 2n + 1 \) be an orthonormal basis of the tangent space at any point of the manifold. Then
putting \( Y = Z = e_i \) in (3.3) and taking summation over \( i, 1 \leq i \leq 2n + 1 \), we get
\[ \sum_{i=1}^{2n+1} (-\nabla WP)(e_i, U) + \frac{\rho_r}{2n} g(X,U) - \frac{\rho_s}{2n} (\nabla WP)(X, \xi)\eta(U) - \frac{\rho_r}{2n} \eta(X)\eta(U) + \sum_{i=1}^{2n+1} g(Y,Z)g(X,U) + 2nB(W)g(X,U). \]
Replacing \( U \) by \( \xi \) in (3.4) and using (2.2)(b) and (2.9), we get
\[ A(W)[(2n + 1) - \frac{r}{2n} \eta(X) + 2nB(W)\eta(X)] = 0. \]

Putting \( X = \xi \) in (3.5), we obtain
\[ B(W) = \frac{r}{4n^2} - \frac{2n+1}{2n}A(W). \]
This leads to the following result:

**Theorem 3.1.** In a generalized projective \( \phi \)-recurrent Sasakian manifold \( M^{2n+1} \), the 1-forms \( A \) and \( B \) are related as in (3.6).

From (3.2) we have,
\[ (\nabla WP)(X,Y)Z = \eta((\nabla WP)(X,Y)Z)\xi - A(W)P(X,Y)Z - B(W)[g(Y,Z)X - g(X,Z)Y]. \]
this implies,
\[ (\nabla WP)(X,Y)Z = \eta((\nabla WP)(X,Y)Z)\xi + \frac{1}{2n} \eta[(\nabla WP)(Y,Z)X - (\nabla WP)(X,Z)Y] \]
\[ + \frac{1}{2n} \eta[\xi] = A(W)[S(Y,Z)X - S(X,Z)Y] \]
\[ + \frac{1}{2n} A(Y)[S(W,Z)\eta(Y) - S(W,Z)\eta(X)] + B(Y)[g(Y,Z)\eta(X) - g(W,Z)\eta(X)]. \]

From (3.8) and the Bianchi identity we get
\[ A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z) \]
\[ = \frac{1}{2n} A(W)[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)] + B(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \]
\[ + \frac{1}{2n} A(X)[S(W,Z)\eta(Y) - S(W,Z)\eta(X)] + B(W)[g(Y,Z)\eta(Y) - g(Y,Z)\eta(W)] \]
\[ + \frac{1}{2n} A(Y)[S(W,Z)\eta(Y) - S(W,Z)\eta(X)] + B(X)[g(W,Z)\eta(Y) - g(W,Z)\eta(X)]. \]

By virtue of (2.8) we obtain from (3.9) that
\[ A(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + A(X)[g(W,Z)\eta(Y) - g(Y,Z)\eta(W)] \]
\[ + A(Y)[g(X,Z)\eta(W) - g(W,Z)\eta(X)] \]
\[ = \frac{1}{2n} A(W)[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)] + B(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \]
\[ + \frac{1}{2n} A(X)[S(W,Z)\eta(Y) - S(W,Z)\eta(X)] + B(X)[g(W,Z)\eta(Y) - g(Y,Z)\eta(W)] \]
\[ + \frac{1}{2n} A(Y)[S(W,Z)\eta(Y) - S(W,Z)\eta(X)] + B(Y)[g(X,Z)\eta(W) - g(W,Z)\eta(X)]. \]

Putting \( Y = Z = e_i \) in (3.10) and taking summation over \( i, 1 \leq i \leq 2n + 1 \), we get
\[ (a) A(W)\eta(X) = A(X)\eta(W) \]
\[ (b) B(W)\eta(X) = B(X)\eta(W). \]
for all vector fields \( X, W \).
Replacing \( X \) by \( \xi \) in (3.11), we get
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(3.12) 
\[ (a) A(W) = \eta(W)\eta(\rho_1) \]
\[ (b) B(W) = \eta(W)\eta(\rho_2), \]
for any vector field $W$, where $A(\xi) = g(\xi, \rho_1) = \eta(\rho_1)$ and $B(\xi) = g(\xi, \rho_2) = \eta(\rho_2)$, $\rho_1$ and $\rho_2$ being the vector fields associated to the 1-forms $A$ and $B$.

From (3.11) and (3.12), we state the following theorem:

**Theorem 3.2.** In a generalized projective $\phi$-recurrent Sasakian manifold $(M^{2n+1}, g)$, $n \geq 1$, the characteristic vector field $\xi$ and the vector fields $\rho_1$ and $\rho_2$ associated to the 1-forms $A$ and $B$ respectively are codirectional and the 1-forms $A$ and $B$ are given by (3.12).

From (2.14) it follows that

(3.13) 
\[ (\nabla_W P)(X,Y)\xi = (\nabla_W R)(X,Y)\xi - \frac{1}{2n} [(\nabla_W S)(Y, \xi)X - (\nabla_W S)(X, \xi)Y]. \]

Using (2.5), (2.6) and (2.9) in the above equation, we have

(3.14) 
\[ (\nabla_W P)(X,Y)\xi = [g(W,\phi Y)X - g(W,\phi X)Y] + R(X,Y)\phi W. \]

By virtue of (2.8) and (2.9) it follows from (3.14) that,

(3.15) 
\[ \eta(\nabla_W P)(X,Y)\xi = 0. \]

Also in a Sasakian manifolds, the following result holds:

(3.16) 
\[ R(X,Y)\phi W = g(\phi X, W)Y - g(Y, W)\phi X - g(\phi Y, W)X + g(X, W)\phi Y + \phi R(X,Y)W. \]

Using (3.14) and (3.16) it follows that

(3.17) 
\[ (\nabla_W P)(X,Y)\xi = g(X, W)\phi Y - g(Y, W)\phi X + \phi R(X,Y)W. \]

In view of (3.14) and (3.16), we obtain from (3.1) that

(3.18) 
\[ g(X, W)\phi Y - g(Y, W)\phi X + \phi R(X,Y)W = -A(W)R(X,Y)\xi - B(W)(g(Y, Z)X - g(X, Z)Y). \]

Using (2.6) and (3.12) in (3.18) we have

(3.19) 
\[ g(X, W)\phi Y - g(Y, W)\phi X + \phi R(X,Y)W = -\eta(W)\eta(\rho_1)\eta(Y)X - \eta(X)Y] - B(W)[\eta(Y)X - \eta(X)Y]. \]

Thus if $X$ and $Y$ are orthogonal to $\xi$, (3.19) reduces to

(3.20) 
\[ \phi R(X,Y)W = g(Y, W)\phi X - g(X, W)\phi Y. \]

Operating $\phi$ on both sides of (3.20) and using (2.1), we get

(3.21) 
\[ R(X,Y)W = g(Y, W)X - g(X, W)Y, \]

for all $X, Y, W$.

Hence we can state the following:

**Theorem 3.3.** A generalized projective $\phi$-recurrent Sasakian manifold $(M^{2n+1}, g)$, $n \geq 1$, is a space of constant curvature, provided that $X$ and $Y$ are orthogonal to $\xi$.

**References**