Some Categorical Aspects of C-Spaces-I

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Abstract. In this paper some simple categorical aspects of c-spaces are attempted. Separators and co-separators are characterized.

I. Introduction

In 1983, Reinhard Börger proposed the theory of connectivity spaces, which ruled out the shortfalls in the use of the theory of connected sets in practical cases. This concept generalized the concept of connectedness in both topology and graph theory. He proved that the category of connectivity spaces \( \mathbb{Z} \) is a topological category which is not cartesian closed. A systematic study of this space was carried out by J. Serra[9] and further extended by H. J. A. M. Heijmans[3], C. Ronse [6] etc. This space found profound applications in the areas of Image Segmentation, Image Filtering, Image Coding, Digital Topology, Pattern Recognition, Mathematical Morphology etc [3, 4, 7, 8, 9]. An initiative towards the mathematical study of the structure of c-spaces can be found in [1, 2, 5]. In this paper we attempt to give a concrete proof to some basic concepts that follows from the concept of Topological Category.

II. Preliminaries

A c-space[5] is a set \( X \) together with a collection \( C \) of subsets such that the following properties hold.

(i) \( \varnothing \in C \) and \( \{x\} \in C \) for every \( x \in X \).

(ii) If \( \{C_i : i \in I\} \) be a non empty collection of subsets in \( C \) with \( \bigcup_{i \in I} C_i \neq \varnothing \) then \( C \) is a connected set.

Other terminologies used for c-spaces are connectivity space[1, 9] and integral connectivity space[2]. In his paper [1], Reinhard Börger have the same definition for the connectivity space except that empty set is connected. The collection \( C \) of subsets \( X \) which satisfies (i) and (ii) is called a c-structure [5] or a connectivity class [3, 4, 9]

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Let \( X \) be a set and \( C \) a collection of subsets. \( \bigcup C \) is called \( C \)-structure. Also, \( C(X) = P(X) \), the power set of \( X \) is \( C \)-structure on \( X \), called the discrete c-space. If \( C \) is a collection of subsets \( X \), called the indiscrete c-space and the corresponding space is called an indiscrete c-space. The c-space \( (X, C_X) \) is denoted by \( X \) if there is no ambiguity.

Let \( X \) and \( Y \) be two c-spaces and \( f : X \to Y \) be a function. \( f \) is called c-continuous [5] or catenuous [5] or a connectivity morphism[3, 9] or a connectivity map[2], if it maps connected sets of \( X \) to connected sets of \( Y \).

A function \( f : X \to Y \) is said to be a quotient map if \( Y \) has the smallest c-structure with respect to which \( f \) is c-continuous. In this context, \( Y \) is said to be the quotient of \( X \) with respect to \( f \).

All categorical terminologies are taken from the text book of Strecker[10]. For definitions, readers are requested to refer the same.

III. On some Basics of the Category of C-Spaces

It can be noted that [1, 2] collection of c-spaces with c-continuous functions as morphisms forms a category. We like to denote the category of c-spaces by \( \text{Cnc} \). By \( \text{Mor}(A, B) \), we mean the collection of c-continuous functions from \( A \) to \( B \). It can be noted that \( \text{Cnc} \) is neither thin nor discrete and not small.
Proposition 3.1. An object \( A \) is an initial object in \( C_{nc} \) if and only if it is the empty \( c \)-space.

Proof. Let \( A \) be an initial object in \( C_{nc} \). Then \( |\text{Mor}(A, B)| = 1 \) for every \( c \)-space \( B \). If \( A \neq \emptyset \), then \( |\text{Mor}(A, B)| > 1 \) for every indiscrete \( c \)-space \( B \) with \( |B| > 1 \), a contradiction to the choice of \( A \). Thus \( A = \emptyset \) and hence \( A \) is an empty \( c \)-space. Conversely let \( A \) be an empty \( c \)-space. Since we can define exactly one function from empty set to any set, it is obvious that \( A \) is an initial object in \( C_{nc} \).

Proposition 3.2. An object \( A \) is a terminal object in \( C_{nc} \) if and only if \( |A| = 1 \).

Proof. Let \( A \) be a terminal object in \( C_{nc} \). Then by the definition, \( |\text{Mor}(B, A)| = 1 \) for every \( c \)-space \( B \). If \( |A| > 1 \), then \( |\text{Mor}(B, A)| > 1 \) for every discrete \( c \)-space \( B \) with \( |B| > 1 \), a contradiction to the choice of \( A \). Hence \( |A| \leq 1 \). Since \( A \) cannot be \( \emptyset \), we have \( |A| = 1 \). Conversely let \( A \) be a \( c \)-space with \( |A| = 1 \). Since \( |A| = 1 \), the only \( c \)-morphism from \( B \) to \( A \) is the constant map. Hence \( |\text{Mor}(B, A)| = 1 \) for every \( c \)-space \( B \).

Thus \( A \) is a terminal object.

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Corollary 3.3. \( C_{nc} \) has no zero objects.

Proposition 3.4. In \( C_{nc} \), the separators are precisely the nonempty \( c \)-spaces.

Proof. Let the \( c \)-spaces \( A \) and \( B \) be such that \( \xrightarrow{f} B \) are distinct \( c \)-morphisms. Then there exists \( a \in A \) such that \( f(a) = g(a) \). Let \( S \) be any nonempty. Define \( h : S \rightarrow A \) by \( h(x) = a \), for all \( x \in S \). Being a constant map, \( h \) is a \( c \)-morphism. It can be easily verified that \( (f \circ h)(x) = f(a) \) and \( (g \circ h)(x) = g(a) \) for all \( x \in S \). Since \( f(a) = g(a) \), we have \( f \circ g = g \circ h \). Hence \( S \) is a separator.

Proposition 3.5. In \( C_{nc} \), indiscrete spaces with at least two points is a co-separator.

Proof. Let \( B \rightarrow A \) be two morphisms such that \( f(b) \neq g(b) \) for some \( b \in B \). Let \( D \) be an indiscrete \( c \)-space with at least two elements \( c_1 \) and \( c_2 \).

Define \( h : A \rightarrow D \) by \( h(x) = c_1 \) if \( x = f(b) \)

Clearly \( h \) is a \( c \)-morphism with \( h \circ f = h \circ g \). Hence \( D \) is a co-separator.

Remark 3.6. The above condition is not necessary.

Proof. Consider the \( c \)-space \( A \) and \( B \) where \( A = \{a, b, c\} \), \( C_A = D_A \cup \{(a, b), (b, c), (a, b, c)\} \), \( B = \{1, 2\} \) and \( C_B = DB \cup \{(1, 2)\} \).

Define \( f : B \rightarrow A \) as \( f(1) = a, f(2) = b \) and \( g : B \rightarrow A \) as \( g(1) = b, g(2) = a \). Here \( f \) and \( g \) are \( c \)-morphisms such that \( f = g \). Choose a \( c \)-space \( E \) with \( E = \{d, e, f\} \) and \( CE = DE \cup \{(d, e), (e, f), (d, e, f)\} \).

Define \( h : A \rightarrow E \) as \( h(a) = d, h(b) = e \) and \( h(c) = f \). Then \( h \) is a \( c \)-morphism with \( h \circ f = h \circ g \). Hence \( E \) is a co-separator. But note that \( E \) is not an co-separator.

Theorem 3.7. Let \( A, B \in C_{nc} \) with \( f, g \in \text{Mor}(B, A) \) with \( f \neq g \). Then \( X \) is a co-separator of \( f \) and \( g \) if and only if \( X \) contains the quotient space of \( A \) with respect to some \( c \)-morphism \( h \in \text{Hom}(A, X) \) with \( h \circ f = h \circ g \).

Proof. Let \( f, g \in \text{Mor}(B, A) \) with \( f \neq g \). Let \( X \) be a co-separator of \( f \) and \( g \). There exists a \( c \)-morphism \( h : A \rightarrow X \) such that \( h \circ f = h \circ g \). Since \( h \) is a \( c \)-morphism, \( CX \) should contain the \( c \)-structure \( C_A^x \Rightarrow \langle h(C) \rangle \) : \( C \in C_A \rangle > . \) Obviously \( (h(A), C_A^x) \) is the quotient space of \( A \) with respect to \( h \).
Conversely let $X$ contains the quotient space of $A$ with respect to some function $h : A \rightarrow X$ with $h \circ f = h \circ g$. Then $h$ is a $c$-morphism with $h \circ f = h \circ g$. $\mathfrak{X}$ is a co-separator of $f$ and $g$.

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**References**


