Iterative Learning Control-Based Optimal Linear Quadratic Digital Tracker for Sampled-Time Active Magnetic Bearing System

Tran Minh Hai\textsuperscript{1,2}, Dang Gia Dung\textsuperscript{2}, Dao Thi My Linh \textsuperscript{2}

\textsuperscript{1}College of Electrical and Information Engineering, Hunan University, ChangSha, Hunan, China. \textsuperscript{2}College of Electrical and Electronic, Thai Binh University, Thai Binh, Viet Nam.

Corresponding Author: Tran Minh Hai

Abstract: This paper presents an iterative learning control (ILC) methodology-based optimal linear quadratic digital tracker (LQDT) for the five degree of freedom (five-DOF) active magnetic bearing (AMB) system as a multi-input multi-output sampled-data system. The combination of ILC and an observer is given out the perfect tracking responses and steady-state. The paper is organized as: (i) System-state observer Kalman filter identification (OKID) is used to construct control system. (ii).

Keywords: Active magnetic bearing (AMB), iterative learning control (ILC), linear quadratic digital tracker (LQDT), five degree of freedom (five-DOF), observer Kalman filter identification (OKID).

I. System-state observer Kalman filter identification (OKID) is used to construct control system

This part is derived as, apply the OKID method to construct the control loop. And after that, ILC is applied to achieve desired goal.

A. Observer Kalman identification algorithm

Basic observer equation

In order to apply the observer Kalman filter identification method, the system state is required to perform as

\[ x(k + 1) = Ax(k) + Bu(k), \]
\[ y(k) = Cx(k) + Du(k), \]

where \( x(k) \in \mathbb{R}^{n \times 1} \), \( y(k) \in \mathbb{R}^{m \times 1} \) and \( u(k) \in \mathbb{R}^{r \times 1} \) are state, output, and control input vectors, respectively, and \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times r} \), \( C \in \mathbb{R}^{m \times n} \) and \( D \in \mathbb{R}^{m \times r} \) are system, input, output and direct transmission term system matrices, respectively.

- Zero initial condition

Assuming zero initial condition \( x(0) = 0 \), for \( k = 0, 1, 2, \cdots, l – 1 \)

\[ x(0) = 0, \]
\[ y(0) = Du(0), \]
\[ x(1) = Bu(0), \]
\[ y(1) = CBu(0) + Du(1), \]
\[ x(2) = ABu(0) + Bu(1), \]
\[ y(2) = CABu(0) + CBu(1) + Du(2), \]
\[
\vdots \\
\]
\[ x(l – 1) = \sum_{i=1}^{l-1} A^{i-1} Bu(l - 1 - i), \]
\[ y(l – 1) = \sum_{i=1}^{l-1} CA^{i-1} Bu(l - 1 - i) + Du(l - 1) \]

DOI: 10.9790/1676-1405012944 www.iosrjournals.org
Iterative Learning Control-Based Optimal Linear Quadratic Digital Tracker …

\( y(k-1) \) for \( k = 1, 2, 3, \ldots, l \)
can be grouped in a matrix form to yield
\( y = YU, \)
where
\[
y = \begin{bmatrix} y(0) & y(1) & y(2) & \cdots & y(l-1) \end{bmatrix},
\]
\( Y = [D \ CB \ CAB \cdots \ CA^{l-2}B], \)
and
\[
U = \begin{bmatrix}
  u(0) & u(1) & u(2) & \cdots & u(l-1) \\
  0 & u(0) & u(1) & \cdots & u(l-2) \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & u(0)
\end{bmatrix},
\]
\( y \in \mathbb{R}^{m \times l}, Y \in \mathbb{R}^{m \times rl}, U \in \mathbb{R}^{rl \times l}. \)

\( m \) is the number of outputs, \( l \) is the number of data samples and \( r \) is the number of inputs. Equation \( y = YU \) is a matrix representation of the relationship between the input and output time histories. Matrix \( Y \) contains all the Markov parameters. Matrix \( U \) is a block upper-triangular input matrix. When the states of system are inaccessible, an observer is usually applied to estimate the states from the information of input and output. Therefore, add and subtract the term \( Gy(k) \), the observer of system can be rewritten as
\[
x(k+1) = Ax(k) + Bu(k) + Gy(k) - Gy(k) \\
= (A + GC)x(k) + (B + GD)u(k) - Gy(k) \\
= \bar{A}x(k) + \bar{B}v(k), \quad (6.2)
\]
where
\[
\bar{A} = A + GC, \quad \bar{B} = [B + GD, -G],
\]
and \( v(k) = \begin{bmatrix} u(k) \\ y(k) \end{bmatrix} \)
and \( G \) is an \( n \times m \) arbitrary matrix that can be used to make the desired stable matrix \( \bar{A} \). In fact, system (6) is an observer equation if the state \( x(k) \) is considered as an observer state vector. Therefore, the Markov parameters of system (1) will be referred to as the observer Markov parameters.

From system (6), it is easy to show that
\[
x(k+1) = \bar{A}x(k) + \bar{B}v(k),
\]
\[
x(k+2) = \bar{A}x(k+1) + \bar{B}v(k+1)
\]
\[
= \bar{A}^2x(k) + \bar{A}\bar{B}v(k) + \bar{B}v(k+1),
\]
\[ \vdots \]
\[
x(k+p) = \bar{A}x(k+p-1) + \bar{B}v(k+p-1)
\]
\[
= \bar{A}^p x(k) + \bar{A}^{p-1} \bar{B}v(k) + \bar{A}^{p-2} \bar{B}v(k+1) + \cdots + \bar{B}v(k+p-1)
\]
and
\[
y(k+p) = Cx(k+p) + Du(k+p)
\]
\[
= CA^p x(k) + CA^{p-1} \bar{B}v(k) + CA^{p-2} \bar{B}v(k+1) + \cdots + C\bar{B}v(k+p-1) + Du(k+p)
\]
for \( k = 0, \ldots, l-1-p \), one has
\[
\bar{y} = CA^p x + \bar{V},
\]

where
\[
\bar{y} = [y(p) \ y(p+1) \ \cdots \ y(l-1)] \in \mathbb{R}^{n \times (l-p)},
\]
\[
x = [x(0) \ x(1) \ \cdots \ x(l-p-1)] \in \mathbb{R}^{m \times (l-p)},
\]
\[
\bar{Y} = [D \ CB \ C\bar{A} \ \cdots \ \bar{C}^{p-1}B | \bar{C}] \in \mathbb{R}^{m \times (r+(r+m)p)}
\]

and
\[
\bar{V} = \begin{bmatrix}
    u(p) & u(p+1) & \cdots & u(l-1) \\
    v(p-1) & v(p) & \cdots & v(l-2) \\
    v(p-2) & v(p-1) & \cdots & v(l-3) \\
    \vdots & \vdots & \ddots & \vdots \\
    v(0) & v(1) & \cdots & v(l-p-1)
\end{bmatrix}
\]
\[
\in \mathbb{R}^{[r+(r+m)p] \times (l-p)}
\]

where \( \bar{A}^p \) is sufficiently small and \( mp \geq n \), so can be approximated by neglecting the first term \( CA^p x \), such that
\[
\bar{y} = \bar{Y}
\]
\[
\bar{y} V^T = \bar{Y} V V^T
\]
\[
\bar{y} V^T (V V^T)^{-1} = Y
\]

- **Nonzero initial condition**

Consider the discrete multivariable linear system described by
\[
x(k+1) = Ax(k) + Bu(k) \quad (7.3a)
\]
\[
y(k) = Cx(k) + Du(k) \quad (7.3b)
\]

Add and subtract the term \( Gy(k) \) to the right-hand side of the state equation, then Eq. (8) can be rewritten by
\[
x(k+1) = Ax(k) + Bu(k) + Gy(k) - Gy(k)
\]
\[
= (A + GC)x(k) + (B + GD)u(k) - Gy(k)
\]
\[
\Rightarrow x(k+1) = \bar{A}x(k) + \bar{B}v(k) \quad (7.4)
\]

For nonzero initial conditions, the above equation is easy to show that
\[
x(k+1) = \bar{A}x(k) + \bar{B}v(k)
\]
\[
x(k+2) = \bar{A}x(k+1) + \bar{B}v(k+1)
\]
\[
= \bar{A}^2 x(k) + \bar{A}\bar{B}v(k) + \bar{B}v(k+1)
\]
\[
\vdots
\]
\[
x(k+p) = \bar{A}x(k+p-1) + \bar{B}v(k+p-1)
\]
\[
= \bar{A}^p x(k) + \bar{A}^{p-1}Bv(k) + \bar{A}^{p-2}Bv(k+1) + \cdots + \bar{B}v(k+p-1)
\]

Using the measurement equation yields
\[
y(k+p) = Cx(k+p) + Du(k+p)
\]
\[
= C\bar{A}^p x(k) + C\bar{A}^{p-1}Bv(k) + C\bar{A}^{p-2}Bv(k+1) + \cdots
\]
\[
+ C\bar{B}v(k+p-1) + Du(k+p)
\]
\[
\Rightarrow \bar{y} = \bar{C} A^p x + \bar{F} V
\]

where the states in \( x \) are bounded and \( \bar{A}^p \) is sufficiently small the above equation can be approximated by
\[
\Rightarrow \bar{y} = \bar{Y} \bar{V}
\]

where
Iterative Learning Control-Based Optimal Linear Quadratic Digital Tracker...

\[ \mathbf{V} = \begin{bmatrix} u(p) & u(p+1) & \cdots & u(l-1) \\ v(p-1) & v(p) & \cdots & v(l-2) \\ v(p-2) & v(p-1) & \cdots & v(l-3) \\ \vdots & \vdots & \ddots & \vdots \\ v(0) & v(1) & \cdots & v(l-p-1) \end{bmatrix} \in \mathbb{R}^{[r+(r+m)p] \times (l-p)} \]

where, all the rows of \( \mathbf{V} \) must be linearly independent. The maximum value of \( p \), which is the upper bound of the order of the deadbeat observer, is the number that maximizes the number \( (r+m)p + r \leq l-p \) of the independent rows of \( \mathbf{V} \). So, \( l \geq (r+m+1)p + r \). On the other hand, the lower bound of \( p \) must be chosen such that \( p \cdot \max(r, m) \geq n \), where \( r \) and \( m \) are the numbers of inputs and outputs, respectively, and \( n \) is the order of the system. Obviously, \( p \) might be smaller than the true order of the system for a multiple-input multiple-output system, while for a single-input single-output system \( p \) must be greater than or equal to the true order of the system.

- Computation of observer Markov parameters

The observer Markov parameters \( \mathbf{V}_k = C\bar{A}^{k-1}B \) include the system Markov parameters \( Y_k = CA^{k-1}B \) and the observer gain Markov parameters \( Y_k^o = CA^{k-1}G \). The system Markov parameters and the observer gain Markov parameters are used to combine a Hankel matrix.

- System Markov Parameters

To recover the system Markov parameters in \( Y \) from the observer Markov parameters in \( \mathbf{V} \), partition \( \mathbf{V} \) such that

\[ \mathbf{V} = \begin{bmatrix} D & CB & C\bar{A}B & \cdots & C\bar{A}^{(p-1)}B \end{bmatrix} \]

\[ = \begin{bmatrix} \mathbf{V}_0 & \mathbf{V}_1 & \mathbf{V}_2 & \cdots & \mathbf{V}_p \end{bmatrix} \]  \hspace{1cm} \text{(7.6)}

where

\[ \mathbf{V}_0 = D. \]

\[ \mathbf{V}_k = C\bar{A}^{k-1}B \]

\[ = \begin{bmatrix} (A+GC)^{k-1}(B+GD) & -C(A+GC)^{k-1}G \end{bmatrix} \]

\[ = \begin{bmatrix} \mathbf{V}_k^{(1)} & -\mathbf{V}_k^{(2)} \end{bmatrix} \]

for \( k = 1, 2, 3, \ldots \).

The system Markov parameters of the system can be reformulated as

\[ Y_1 = CB = C(B+GD)-(CG)D = \mathbf{V}_1^{(1)} - \mathbf{V}_1^{(2)}D. \]

\[ \mathbf{V}_2^{(1)} = C(A+GC)(B+GD) \]

\[ = CAB + CGCB + C(A+GC)GD \]

\[ = Y_2 + \mathbf{V}_1^{(2)}Y_1 + \mathbf{V}_2^{(2)}D \]

\[ Y_2 = CAB = \mathbf{V}_2^{(1)} - \mathbf{V}_2^{(2)}Y_2 - \mathbf{V}_2^{(2)}D. \]
\[ Y_3^{(1)} = C(A + GC)^2(B + GD) \]
\[ = C(A^2 + GCA + AGC + GCGC)(B + GD) \]
\[ = CA^2B + CGCAB + C(A + GC)GCB + C(A + GC)^2GD \]
\[ = Y_3 + \bar{r}_1^{(2)}Y_2 + \bar{r}_2^{(2)}Y_1 + \bar{r}_3^{(2)}D. \]

Then, one has
\[ Y_3 = CA^2B = Y_3^{(1)} - \bar{r}_1^{(2)}Y_2 - \bar{r}_2^{(2)}Y_1 - \bar{r}_3^{(2)}D. \]

By induction, the general relationship between the system Markov parameters \( Y_k \) and the observer Markov parameters \( \bar{Y}_k \) is
\[ Y_0 = \bar{Y}_0 = D, \]
\[ Y_k = \bar{Y}_k^{(1)} - \sum_{i=1}^{k} \bar{r}_i^{(2)}Y_{k-i} \text{ for } k = 1, 2, \ldots, p. \]
\[ Y_k = - \sum_{i=1}^{p} \bar{r}_i^{(2)}Y_{k-i} \text{ for } k = p + 1, \ldots, \infty. \]

- **Observer Gain Markov Parameters**

To identify the observer gain \( G \), first recovers the sequence of parameters as follows:
\[ Y_k^o = CA^{k-1}G \text{ for } k = 1, 2, 3, \ldots. \]

In terms of the observer gain Markov parameters, in fact, the first parameter of equation in the sequence is
\[ Y_1^o = CG = \bar{Y}_1^{(2)}. \]

The next parameter in the sequence is obtained by considering \( \bar{Y}_2^{(2)} \)
\[ \bar{Y}_2^{(2)} = C\bar{A}G = (CAG + CGCG) \]
\[ = Y_2^o + \bar{r}_1^{(2)}Y_1^o. \]

Then, one has
\[ Y_2^o = \bar{r}_2^{(2)}Y_1^o - \bar{r}_1^{(2)}Y_1^o. \]

Similarly, one gets
\[ \bar{Y}_3^{(2)} = C\bar{A}^2G \]
\[ = (CA^2G + CGCAG + C\bar{A}CGC) \cdot \]
\[ = Y_3^o + \bar{r}_1^{(2)}Y_2^o + \bar{r}_2^{(2)}Y_1^o \]

Then has
\[ Y_3^o = \bar{r}_3^{(2)}Y_2^o - \bar{r}_1^{(2)}Y_2^o - \bar{r}_2^{(2)}Y_1^o. \]

The general relationship can be summarized as follows:
\[ Y_1^o = CG = \bar{Y}_1^{(2)}, \]
\[ Y_k^o = \bar{r}_k^{(2)} - \sum_{i=1}^{k-1} \bar{r}_i^{(2)}Y_{k-i}^o. \]
Iterative Learning Control-Based Optimal Linear Quadratic Digital Tracker

for $k = 2, 3, \cdots, p$,

$$Y^O_k = -\sum_{i=1}^{p} \tilde{r}^{(2)}_i Y^O_{k-i}$$

for $k = p + 1, \cdots, \infty$.

- **Eigensystem realization algorithm**

  The Hankel matrix $\hat{H}(k-1)$ from the combined observer Markov parameters is associated with the system and observer as

  $$\hat{H}(k-1) = \begin{bmatrix} Y_k & Y_{k+1} & \cdots & Y_{k+\beta-1} \\ Y_{k+1} & Y_{k+2} & \cdots & Y_{k+\beta} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{k+\alpha-1} & Y_{k+\alpha} & \cdots & Y_{k+\alpha+\beta-2} \end{bmatrix}$$

  where $\hat{H}(k-1)_{(m \times \alpha) \times (\beta \times (m+r))}$ with $\alpha \geq 0$ and $\beta \geq 0$ are sufficiently large arbitrary integers and

  $$Y_k = [Y_k^{(2)} \ Y_k^{(1)}] = [CA^{k-1}B \ CA^{k-1}G].$$

  Notice that it’s required $(m \times \alpha) < [\beta \times (m+r)]$ and $\alpha \geq p$. However, a large $\alpha$ may induce a large numerical computation error. When the combined observer Markov parameters are determined, the eigen-system realization algorithm (ERA) method is used to obtain the desired discrete system realization $[A, B, C, G]$ through singular value decomposition (SVD) of the Hankel matrix.

  The ERA processes the factorization of the block data matrix, started for $k = 1$, using the singular value decomposition $\hat{H}(0) = R \Sigma S^T$, where the columns of matrices $R$ and $S$ are orthonormal and $\Sigma$ is a rectangular matrix of the form as follows

  $$\Sigma = \begin{bmatrix} \sum_{\tilde{n}} & 0 \\ 0 & 0 \end{bmatrix},$$

  where $\sum_{\tilde{n}} = diag[\sigma_1, \sigma_2, \cdots, \sigma_{n_{\min}}, \sigma_{n_{\min}} + 1, \cdots, \sigma_{\tilde{n}}]$ contains monotonically non-increasing entries $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n_{\min}} \geq \sigma_{n_{\min}} + 1 \geq \cdots \geq \sigma_{\tilde{n}} > 0$. Here, some singular values $(\sigma_{n_{\min}} + 1, \cdots, \sigma_{\tilde{n}})$ are relatively small $(\sigma_{n_{\min}} + 1 \ll \sigma_{\tilde{n}})$ and negligible in the sense that they contain more noise information than system information. In order to construct the low order observer of the system, let’s define $\sum_{\min} = diag[\sigma_1, \sigma_2, \cdots, \sigma_{n_{\min}}]$. In other words, the reduced model of order $n_{\min}$ after deleting singular values $(\sigma_{n_{\min}} + 1, \cdots, \sigma_{\tilde{n}})$ is then considered as the robustly controllable and observable part of the realized system with an acceptable closed-loop performance. Simultaneous realizations of the system and observer by the ERA are given as

  $$\hat{H}(0) = R \Sigma S^T = [R \sum_{\min}^1/2 \Sigma_{\min}^{1/2} S^T] = \{P\} \{Q\}$$

  where

  $$P = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\alpha-1} \end{bmatrix}$$

  and

  $$Q = \begin{bmatrix} B & AB & \cdots & A^{\beta-1}B \end{bmatrix}.\]
then
\[
A = \sum_{n=1}^{N} V_n^{T} \min_{\mathbf{min}} H(l) S_n \min_{\mathbf{min}} \sum_{n=1}^{N} V_n^{T},
\]
\[
[C] = \text{First } (r + m) \text{ columns of } \sum_{n=1}^{N} \min_{\mathbf{min}} S_n^{T}
\]

In order to more improve the performance, this paper used the iterative learning control to achieve this goal. This proposed method use \( \alpha = 2 \), and \( \beta = 3 \)

### B. Iterative learning control

This part will leads us go through an optimal tool, herein distances tracking error of AMB system are reduced, and steady-state is improved. The propose tool based on learning by the past active of some tasks in the system. In order to achieve the high performance and keep the advantage of the high gain system for sampled-data AMB system.

Consider the discrete-time system minimum phase as

\[
\begin{align*}
\dot{x}_{cj}(kT) &= A_{c}x_{cj}(kT) + Bu_{cj}(kT) + d(kT) \\
y_{cj}(kT) &= C_{c}x_{cj}(kT) + Du_{cj}(kT) + s(kT)
\end{align*}
\]

where \( x_{cj}(0) = x(0) \), and \( j \) is iteration number, \( d(kT) \), \( s(kT) \) are unknown disturbances. Then can imagine there exists an equivalently artificial system model.

\[
\begin{align*}
\dot{x}_{a}(kT) &= A_{a}x_{a}(kT) + Bu_{a}(kT) \\
y_{a}(kT) &= C_{a}x_{a}(kT) + Du_{a}(kT) + s_{a}(kT)
\end{align*}
\]

\( s_{a}(kT) \) denotes to actual steady-state error signal between the actual output of the system \( y_{cj}(t) \) and pre-specified trajectory signal. Where the artificial control can be determined as

\[
u_{a}(kT) = -K_{a}x_{a}(kT) + S_{a}r(kT) + C_{a}(kT) + C_{u_{a}}u^{*}_{a}(kT)
\]

where

\[
K_{a} = R_{a}^{-1}(B_{a}^{T} P_{a} + N_{a}^{T}),
\]

\[
S_{a} = -R_{a}^{-1} \left[(C - DK_{a})(A - BK_{a})^{-1}B - D \right]^{T} Q_{a}.
\]

\[
C_{a}(kT) = -S_{a}s_{a}(kT),
\]

\[
C_{a}^{*}(kT) = -a^{-1}(I_{m} + B^{T} [(A - B)]^{-1} K_{a}^{T})R_{a},
\]

\[
N_{a} = C^{T} Q_{a} D,
\]

\[
R_{a} = R_{a} + D^{T} Q_{a} D,
\]

Then \( s_{a}(kT) \) is determined as

\[
s_{a}(kT) = \sum_{i=0}^{j} e_{i-1}(kT)
\]

Where \( e_{i-1}(kT) = y_{c(i-1)}(kT) - r(kT) \),

Then apply optimal error compensation ILC leads to

\[
u_{d}(kT) = -K_{d}x_{d}(kT) + S_{d}r(kT) + C_{d}(kT) + C_{u_{d}}u^{*}_{d}(kT)
\]
An optimal LQDT with pre-specified measurement output and control input trajectories for the discrete-time controllable and observable system with both an input-to-output direct-feedthrough term and known system disturbances is summarized as follows [16].

Consider the controllable and observable linear discrete-time system with an input-to-output direct-feedthrough term and known/estimated system disturbances $d(k)$ and $s(k)$

$$x_d(k+1) = G x_d(k) + H u_d(k) + d(k), \quad (7.8a)$$

$$y_d(k) = C x_d(k) + D u_d(k) + s(k), \quad (7.9b)$$

where $G \in \mathbb{R}^{n \times n}$, $H \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, and $D \in \mathbb{R}^{p \times m}$ are state, input, output, and direct-feedthrough matrices, respectively. $x_d(k) \in \mathbb{R}^n$ is the state vector, $u_d(k) \in \mathbb{R}^m$ is the control input, and $y_d(k) \in \mathbb{R}^p$ is the measurable output. The design goal is to determine the optimal control sequence $u_d(0)$, $u_d(1)$, $u_d(2)$, $\cdots$, $u_d(N_f-1)$ that minimizes the linear quadratic performance index for a finite time process

$$J(x_d, u_d) = \frac{1}{2} \left[ y_d(N_f) - r(N_f) \right]^T S \left[ y_d(N_f) - r(N_f) \right] +$$

$$\sum_{k=0}^{N_f-1} \left[ y_d(k) - r(k) \right]^T Q_d \left[ y_d(k) - r(k) + u_d(k) - u^*_d(k) \right] + u_d(k) - u^*_d(k)^T R_d [u_d(k) - u^*_d(k)] (14)$$

where $Q_d$ is a $p \times p$ positive definite or positive semi-definite real symmetric matrix, $R_d$ is an $m \times m$ positive definite real symmetric matrix, $S$ is a $p \times p$ positive definite or positive semi-definite real symmetric matrix, $r(k)$ is a pre-specified output trajectory, and $u^*_d(k)$ is a pre-specified input trajectory. The resulting continuous-time state-feedback control law is given by

$$u_d(k) = -K_d x_d(k) + E_d r(k) + C_d(k) + C u^*_d(k), \quad (7.10)$$

where

$$K_d = \bar{R}_d^{-1} P_d, \quad E_d = \bar{R}_d^{-1} \left[ D^T + H^T \left[ I_p - \left( G - HK_d \right)^T \right]^{-1} \left( C - DK_d \right)^T \right] Q_d,$$

$$S_d = -\bar{R}_d^{-1} \left[ (C - DK_d) (A - BK_d)^{-1} B - D \right]^T Q_d,$$

$$C_d(kT) = -S_d e_d(kT),$$

$$C_d^* = -\bar{R}_d^{-1} (I_m + B^T [(A - B)]^{-1} K_d^T) R_d,$$

$$N = C^T Q_d D,$$

$$\bar{R}_d = R_d + D^T Q_d D,$$
\[
C_d(k) = \tilde{R}_d^{-1} \left[ H^T \left[ \left( G - HK_d \right)^T - I_n \right]^{-1} \left( C - DK_d \right)^T - D^T \right] Q_d s(k) \\
+ \tilde{R}_d^{-1} H^T \left[ \left( G - HK_d \right)^T - I_n \right]^{-1} (G - HK_d)^T - I_n \right]^{T} P d(k) \\
- E_d s(k) + Z d(k), \\
Z_d = \tilde{R}_d^{-1} H^T \left[ \left( G - HK_d \right)^T - I_n \right]^{-1} (G - HK_d)^T - I_n \right] P \\
C_u^* = \tilde{R}_d^{-1} \left[ H^T \left[ \left( G - HK_d \right)^T - I_n \right]^{-1} K_d^{T} + I_m \right] R_d. \\
\]

\[
\tilde{R}_d = R_d + D^T Q_d D, \\
N_d = C^T Q_d D, \\
\tilde{R}_d^{*} = \tilde{R}_d + H^T PH, \\
\bar{P} = H^T PG + N_d^T, \\
\]
and \( P \) satisfies the algebraic Riccati equation
\[
P = G^T PG + C^T Q_d C - \bar{P} \tilde{R}_d^{-1} \bar{P} \\
= G^T PG + C^T Q_d C - \left( H^T PG + N_d^T \right)^T \left[ \tilde{R}_d + H^T PH \right]^{-1} \left( H^T PG + N_d^T \right). (7.11) \\
\]

C. Sampled-data controlled system
Let the corresponding digitally controlled model of (x) be described as
\[
\dot{x}_d(t) = A x_d(t) + Bu_d(t) + d(t), \quad x_d(0) = x_0 \\
y_d(t) = C x_d(t) + s(t), \\
\]
where \( u_d(t) \in \mathbb{R}^m \) is piecewise-constant, such that \( u_d(t) = u_d(kT), \) for \( kT < t < (k+1)T, \) and \( T > 0, \) is the period of sampling and hold. Let \( u_d(t) \) be a discrete-time state-feedback control law of the form
\[
u_d(kT) = -K_d x_d(kT) + E_d r^*(kT) + C_d(kT) + C_u^* u_d^*(kT), \\
kT < t < (k+1)T, \\
\]
where \( K_d \in \mathbb{R}^{m \times n} \) and \( E_d \in \mathbb{R}^{m \times p} \) are the feedback and feed-forward digital gains, respectively, \( C_d(kT) \) is the compensatory signal, and \( r^*(kT) \) is a piecewise-constant reference input vector to be determined in terms of \( r(kT) \) for tracking purpose. The digital reference input vector with tracking purpose is specified as \( r^*(k) = r(k+1). \) For the direct-feedthrough term-free case. The viewpoint has been proved in \([18-21]\). The overall digitally controlled closed-loop system becomes
\[
\dot{x}_d(t) = Ax_d(t) + B[-K_d x_d(kT) + E_d d(kT) + C_d(kT) + C_u u_d(kT)],
\]

\[
x_d(0) = x_0,
\]

for \( kT < t < (k+1)T \), where the controller is realized using a zero-order-hold, the discrete-time model is described as

\[
x_d((k+1)T) = G x_d(kT) + H u_d(kT) + H_d d(kT),
\]

\[
y_d(kT) = C x_d(kT) + s(kT)
\]

where

\[
G = e^{AT},
\]

\[
H = (G-I)^{-1} A^{-1} B, \text{ if } A^{-1} \text{ exists}
\]

Or

\[
H = \sum_{i=0}^{\infty} \frac{T^{i+1}}{(i+1)!} A^i B, \text{ if } A^{-1} \text{ does not exist, and}
\]

\[
H_d = (G-I)^{-1} A^{-1} I_n, \text{ if } A^{-1} \text{ exists}
\]

Then the control system is constructed as follows

![Diagram](Figure 2.1 Iterative learning control for five DOF AMB system)
Figure 2.2 The actual output responses, (a) y1(t) (b) y2(t) (c) y3(t) (d) y4(t)
Figure 2.3 System orbit, (a) left rotor orbit (b) right rotor orbit
Figure 2.4 control signals, (a) $i_b \pm i_{x1}$, (b) $i_b \pm i_{y1}$, and (c) $i_b + i_z$. 

![Graph](image-url)
Figure 2.5 Open-loop step response

Figure 2.6 Closed-loop step response

Figure 2.7 the z-axis response
II. Summary

This study proposed the quite good controller for sampled-data active magnetic bearing system. To achieve the specified goal, this paper has equipped some advanced techniques such as state observer, Kalman filter to estimate the system state, use iterative learning control tool to reduce tracking error and develop the transient responses signal, and based on the linear quadratic digital tracker is built as the background of this proposed controller. Then, simulation results are given to demonstrate how effective the proposed methodology is on the system.
## Table 3.1: The system parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$</td>
<td>The mass of the rotor</td>
<td>2.56478</td>
<td>kg</td>
</tr>
<tr>
<td>$L$</td>
<td>The length of the rotor</td>
<td>0.505</td>
<td>m</td>
</tr>
<tr>
<td>$\delta$</td>
<td>diameter of rotor</td>
<td>0.0166</td>
<td>M</td>
</tr>
<tr>
<td>$J$</td>
<td>The coefficient of inertia of rotor about X-Y axes</td>
<td>4.004e-2</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>$J_z$</td>
<td>The polar mass moment of inertia of rotor about Z-axis</td>
<td>6.565e-4</td>
<td>kg m$^2$</td>
</tr>
<tr>
<td>$k_{ri}$</td>
<td>The current stiffness of the RAMB</td>
<td>80</td>
<td>N/A</td>
</tr>
<tr>
<td>$k_{rp}$</td>
<td>The position stiffness of the RAMB</td>
<td>2.2e5</td>
<td>N/m</td>
</tr>
<tr>
<td>$k_{ai}$</td>
<td>The current stiffness of the TAMB</td>
<td>40</td>
<td>N/A</td>
</tr>
<tr>
<td>$k_{ap}$</td>
<td>The position stiffness of the AMB</td>
<td>3.6e4</td>
<td>N/m</td>
</tr>
<tr>
<td>$a$</td>
<td>The distance between CG and left RAMB</td>
<td>0.160</td>
<td>m</td>
</tr>
<tr>
<td>$b$</td>
<td>The distance between CG and right RAMB</td>
<td>0.190</td>
<td>m</td>
</tr>
<tr>
<td>$c$</td>
<td>The distance between CG and external disturbances</td>
<td>0.263</td>
<td>m</td>
</tr>
<tr>
<td>$x_b, y_b$</td>
<td>The nominal air gaps in X-Y axes of RAMB</td>
<td>0.4</td>
<td>mm</td>
</tr>
<tr>
<td>$z_0$</td>
<td>The nominal air gap in Z-axis of TAMB</td>
<td>0.5</td>
<td>mm</td>
</tr>
</tbody>
</table>