IOSR Journal of Computer Engineering (IOSR-JCE)
e-ISSN: 2278-0661, p-ISSN: 2278-8727, Volume 27, Issue 6, Ser. 2 (Nov. – Dec. 2025), PP 06-18
www.iosrjournals.org

Reliability Estimation of the Three-Parameter Gumbel Distribution Using Lindley's Approximation under the Bayesian Framework

[1]Jyoti pandey and [2]Prof. Uma Srivastava
[1]Research scholar, Department of Mathematics and Statistics, Deen Dayal Upadhyay Gorakhpur University, Gorakhpur
[2]Professor, Department of Mathematics and Statistics, Deen Dayal Upadhyay Gorakhpur University, Gorakhpur

Abstract
Objectives:
Three key Gumbel distribution parameters—shape (α), scale (β), and position (γ)—are the main focus of this work. In reliability analysis, these characteristics aid in characterising severe circumstances such very high stress levels or lengthy failure durations. The primary goal is to compare two estimate techniques: the Approximate Bayesian estimate (ABS), which is derived using Lindley's approximation under the cautious loss function, and the conventional Maximum Likelihood Estimation (MLE).
Methods:
Simulation experiments were conducted using different sample sizes ranging from (10 to 80), with each case repeated 500 times. In the Bayesian approach, a joint prior distribution was considered with hyperparameters (ξ = 5) and (δ = 1.5). Both the MLE and ABS methods were used to estimate the parameters (α), (β), and (γ), and their performance was evaluated based on the mean estimates and the Mean Squared Errors (MSE).
Findings:
The simulation results show that the ABS method usually performs better than the MLE method, especially when the sample size is small. It gives more stable and accurate parameter estimates with lower MSE values. As in the increase of sample size, both methods become more accurate and start approaching towards the true parameter values, but ABS still provides more consistent and reliable results.
Novelty:
This study highlights the effectiveness of the Bayesian estimation approach using Lindley's approximation under the precautionary loss function for estimating the parameters of the three-parameter Gumbel distribution. Unlike earlier studies that relied on traditional methods, this study applies the Bayesian approach to obtain precise results and more stable, particularly for lesser sample sizes. The results help in reliability analysis by giving a simple, accurate, and fast way to study extreme data and estimate the parameters correctly.
Keywords:
Three-parameter Gumbel distribution; Reliability estimation; Maximum Likelihood Estimation (MLE); Approximate Bayesian Estimation (ABS); Lindley's approximation

---

Date of Submission: 12-11-2025 Date of Acceptance: 24-11-2025

I. Introduction

Reliability means how likely it is that a system or component will work properly and not fail during a specific period of time. It is represented mathematically as R(t)=1-F(t), where F(t) is the cumulative distribution function that shows the likelihood of failure prior to time t. The three parameters of interest are the shape (α), scale (β), and location (γ). In engineering and industry, reliability analysis is essential for designing safe, durable, and cost-effective products while minimizing unexpected failures.

The three-parameter Gumbel distribution is a useful tool for studying extreme events, like the highest stress a material can handle, the longest time before something breaks, or very strong weather conditions. It can be hard to find its exact values, especially when you don't have much data. Common techniques such as Maximum Likelihood Estimation (MLE) often give unreliable or biased results with small datasets. Because of this, researchers are working to create better methods that can still give good results even when the data are limited or uncertain.

Reliability analysis has been studied extensively. Sinha [1] provided a comprehensive overview of reliability and life testing methods. Barlow and Proschan [3] provided the mathematical foundation for reliability modelling, while Kotz and Nadarajah [4] and Coles [5] highlighted the importance of extreme-value distributions in analysing maximum lifetimes. O'Connor and Kleyner [6] and Elsayed [7] linked these theoretical models to industrial applications, highlighting the need for accurate lifetime estimation. Despite these valuable contributions, most classical approaches depend on heavily on large sample sizes and do not perform well when data show small-sample variability or asymmetrical behaviour.

The Bayesian framework offers a more flexible alternative. Bernardo and Smith [8] provided a detailed account of Bayesian inference, and Lindley [2] introduced an approximation method that simplifies posterior computation. However, most applications of Lindley's approximation assume symmetric loss functions such as the Squared Error Loss Function (SELF), which may not accurately reflect real-world reliability situations where underestimation of failure probability is more serious than overestimate. Srivastava and Yadav [10] applied approximate Bayesian estimation under a precautionary (asymmetric) loss function to the generalized compound Rayleigh distribution, demonstrating that such loss structures yield more meaningful parameter estimates. However, their approach cannot be directly applied to the extreme-value behaviour of the Gumbel model. Similarly, Okumu et al. [11] developed estimation techniques for the three-parameter Gumbel distribution but did not integrate Bayesian approximation with asymmetric loss functions.

From this review, a clear research gap emerges: existing literature has not fully explored the integration of Lindley's approximation with asymmetric (precautionary) loss functions on the three-parameter of Gumbel distribution. This limitation reduces robustness of existing estimation frameworks, particularly in reliability analysis involving small or extreme datasets.

To address this gap, the present study proposes an Approximate Bayesian Estimation (ABS) framework based on Lindley's approximation under a precautionary loss function. The proposed method introduces asymmetry in the loss structure, penalizing underestimation more heavily to reflect real-world reliability risks. Compared to MLE and Bayesian estimators under symmetric loss, the new method improves estimation accuracy, reduces bias for small samples, and remains computationally efficient. Simulation experiments conducted across multiple sample sizes confirm that the proposed ABS approach provides more stable and precise parameter estimates than MLE, establishing it as a superior tool for the reliability estimation of the three-parameter Gumbel distribution [11–14].

II. The Gumbel Distribution

Assume that x is a random variable that represents the observations or data points. In real-world applications, it could stand for extreme values like highest temperatures, flood levels, or monetary losses. Let f(x) be the value of the probability density function at x.

$$F(x;\ \alpha,\beta,\gamma) = \begin{cases} \frac{1}{\beta}\ e^{-\left(\frac{x-\gamma-\alpha}{\beta}\right)}\ e^{-e^{-\left(\frac{x-\gamma-\alpha}{\beta}\right)}}, & x > \gamma \\ 0, & x \le \gamma \end{cases} \qquad (1.1)$$

where the distribution's position is determined by the location parameter, γ.

The scale parameter, $\beta > 0$, regulates the distribution's spread.

The shape parameter is represented by α.Because the Gumbel distribution is not specified below this threshold when employing a shift form, the pdf is 0 for $x \le \gamma$.

The exponential term is followed by the pdf for $x > \gamma$.

The probability density function may be integrated to get the cumulative distribution function for the three-parameter Gumbel distribution.

$$F(x) = e^{-e^{-\left(\frac{x-\gamma-\alpha}{\beta}\right)}} \qquad (2.1)$$

Reliability function r(t) for the three-parameters Gumbel distribution

$$R(t) = 1 - F(t) \qquad (3.0)$$

$$= 1 - e^{-e^{-\beta(t-\gamma-\alpha)}}$$

Compute log R

$R(t) = 1 - e^{-e^{-\beta(t-\gamma-\alpha)}}$

Let z= $\beta(t-\gamma-\alpha)$

R(t)= $1 - e^{-e^{-z}}$

Log R(t)= $log\left(1 - e^{-e^{-p}}\right)$

$$= log\ \left(1-\ e^{-e^{-\beta(t-\gamma-\alpha)}}\right) \quad (3.1)$$

β in terms of R

$$R = 1-\ e^{-e^{-\beta(t-\gamma-\alpha)}}$$
$$e^{-e^{-\beta(t-\gamma-\alpha)}} = 1-R$$
$$-e^{-\beta(t-\gamma-\alpha)} = log\ (1-R)$$
$$-\beta(t-\gamma-\alpha) = log\ (-log(1-R))$$
$$\beta = -\frac{1}{(t-\gamma-\alpha)}\ log\ (-log(1-R)) \quad (3.2)$$
$$\frac{d\beta}{dR} = \ -\frac{1}{(t-\gamma-\alpha)}\ \frac{1}{(1-R)(-log(1-R))} \quad (3.3)$$

Then we take the posterior.

$$h(\beta|\ \underline{x}) = \frac{\prod_{i=1}^{n}[\beta\ e^{(-\beta(x-\gamma-\alpha))}e^{-e^{(-\beta(x-\gamma-\alpha))}}]\ \frac{b^a}{\Gamma a}\ \beta^{a-1}\ e^{-\beta b}}{\int_0^\infty \prod_{i=1}^{n}[\beta\ e^{(-\beta(x-\gamma-\alpha))}e^{-e^{(-\beta(x-\gamma-\alpha))}}]\ \frac{b^a}{\Gamma a}\ \beta^{a-1}\ e^{-\beta b}\ d\beta}$$

Represents the posterior distribution of the parameter β.

It is obtained by combining the likelihood function of the three-parameter Gumbel distribution with a gamma prior for β.

Here,

- $\prod_{i=1}^{n} \beta e^{(-\beta(x-\gamma-\alpha))}e^{-e^{-(\beta(x-\gamma-\alpha))}}$ is the likelihood
- $G(\beta|\underline{x}) = \frac{b^a}{\Gamma a}\ \beta^{a-1}\ e^{-\beta b}$ *is* the Gamma prior for β

and the denominator is the normalizing constant that ensures the posterior integrates to 1.
Thus, h(β|x) represents the Bayesian posterior of β based on the observed sample x.

$$h(R(t)) = \frac{(u+b)^{n+a+1}}{\Gamma(n+a+1)}\left(\frac{-1}{(t-\gamma-\alpha)}\ log\ (-log(1-R))^{(n+a+1)}\right) e^{\frac{u+b}{(t-\gamma-\alpha)}log\ (-log(1-R))}\ d\beta$$

$$= \frac{(u+b)^{n+a+1}}{\Gamma(n+a+1)}\left(\frac{-1}{(t-\gamma-\alpha)}\ log\ (-log(1-R))^{(n+a+1)}\right) e^{\frac{u+b}{(t-\gamma-\alpha)}log\ (-log(1-R))}\ \frac{1}{(t-\gamma-\alpha)}\ \frac{1}{(1-R)(-log(1-R))}$$

$$h(R(t)) = \frac{(u+b)^{n+a+1}}{\Gamma(n+a+1)}\left(\frac{-1}{(t-\gamma-\alpha)}\ log\ (-\ log(1-R))\right)^{(n+a+1)} e^{\frac{u+b}{(t-\gamma-\alpha)}log\ (-log(1-R))}\ \frac{1}{(t-\gamma-\alpha)}\ \frac{1}{(1-R)(-log(1-R))}$$

(4.1)

let u = (x-γ-α) and $p= -\log(1-R)$

So, $R= 1-e^{-p}$

$$h(R(t)) = \frac{(u+b)^{n+a+1}}{\Gamma(n+a+1)}\left(\frac{-1}{(t-\gamma-\alpha)}\ log\ p\right)^{(n+a+1)} e^{\frac{u+b}{(t-\gamma-\alpha)}log\ p}\ \frac{1}{(t-\gamma-\alpha)\ e^{-p}(-p)} \quad (4.2)$$

This is the posterior density function of the three-parameter Gumbel distribution's reliability function.

Bayes estimator under the precautionary loss function for the Reliability function.

$$\hat{R}_{BP} = \left[E_\rho(R^2)\right]^{\frac{1}{2}} \quad (5.1)$$

$$= \int_0^1 R^2 h(R(t))\ dR \quad (5.2)$$

We know that, $p = -\log(1-R)$, and $R = 1-e^{-p}\ as\ R \to 1, p \to \infty\ and\ R \to 0 then$
$p \to 0$

$R = 1-e^{-p} \quad then\ R^2 = (1-e^{-p})^2$

$dR = e^{-p}\ dp$

Putting all this value in equation (5.2)

$$= \int_0^1 R^2 \frac{(u+b)^{n+a+1}}{\Gamma(n+a+1)}\left(\frac{-1}{(t-\gamma-\alpha)}\ log\ p\right)^{(n+a+1)} e^{\frac{u+b}{(t-\gamma-\alpha)}log\ p}\ \frac{1}{(t-\gamma-\alpha)\ e^{-p}(-p)} e^{-p}\ dp \quad (5.3)$$

Simplify the above expression

$$e^{\frac{u+b}{(t-\gamma-\alpha)}log\ p} = p^{\frac{u+b}{(t-\gamma-\alpha)}}$$

And let $\lambda = \frac{u+b}{(t-\gamma-\alpha)}$

$$C = \frac{(u+b)^{n+a+1}}{\Gamma(n+a+1)}\left(\frac{1}{(t-\gamma-\alpha)}\right)^{(n+a+2)}$$

Now the final integral becomes

$$C\int_0^\infty (1-e^{-p})^2 p^{\lambda-1} \log p^{n+a+1}\ dp \quad (5.4)$$

Then after solving the equation (5.4) we obtain

$$=\frac{(u+b)^{n+a+1}}{\Gamma(n+a+1)}\left(\frac{1}{(t-\gamma-\alpha)}\right)^{(n+a+2)}\frac{1}{2\left(\frac{u+b}{t-\gamma-\alpha}\right)}-1\,\Gamma(n+a+1)\quad\left(\frac{u+b}{t-\gamma-\alpha}\right) \qquad (5.5)$$

$$\widehat{R}_{BP}=\left(\frac{(u+b)^{n+a+2}}{(t-\gamma-\alpha)^{n+a+2}\left(2(u+b)-(t-\gamma-\alpha)\right)}\right) \qquad (5.6)$$

The Bayesian approach to estimating the unknown parameters α,β and γ

The combined prior distribution corresponding to the parameters α,β, and γ can be expressed as

$J(\alpha,\beta,\gamma)=j_1(\alpha)j_2(\gamma)j_3(\beta|\gamma)$

Taking

$$j_1(\alpha)=c \qquad (6.1)$$

$$j_2(\gamma)=\frac{1}{\delta}e^{-\frac{\gamma}{\delta}} \qquad (6.2)$$

$$j_3(\beta|\gamma)=\frac{1}{\Gamma\xi}\gamma^{-\xi}\beta^{\xi+1}e^{\left[-\frac{\beta}{\gamma}\right]} \qquad (6.3)$$

$$J(\alpha,\beta,\gamma)=j_1(\alpha)j_2(\gamma)j_3(\beta|\gamma)$$

$$=\frac{c}{\delta\Gamma\xi}\gamma^{-\xi}\beta^{\xi+1}e^{\left[-\left(\frac{\gamma}{\delta}+\frac{\beta}{\gamma}\right)\right]} \qquad (6.4)$$

Joint posterior with likelihood equation number (3.1) and (6.4) we get;

$$h^*(\alpha,\beta,\gamma)=\frac{\gamma^{-\xi}\beta^{\xi+1}e^{\left[-\left(\frac{\gamma}{\delta}+\frac{\beta}{\gamma}\right)\right]}L(\underline{x}|\alpha,\beta,\gamma)}{\iiint\gamma^{-\xi}\beta^{\xi+1}e^{\left[-\left(\frac{\gamma}{\delta}+\frac{\beta}{\gamma}\right)\right]}L(\underline{x}|\alpha,\beta,\gamma)\,d\alpha d\beta d\gamma} \qquad (6.5)$$

The approximate Bayes estimators

$V(\Theta)=v(\alpha,\beta,\gamma)$

$$\hat{v}_{AB}=E(v|\underline{x})=\frac{\iiint v(\alpha,\beta,\gamma)J^*(\alpha,\beta,\gamma)\,d\alpha d\beta d\gamma}{\iiint J^*(\alpha,\beta,\gamma)\,d\alpha d\beta d\gamma} \qquad (6.6)$$

2.1 Lindley Approximation

$E(v(\alpha,\beta,\gamma|\underline{x}))$

$$=V(\Theta)+\frac{1}{2}[S(v_1\sigma_{11}(v_2\sigma_{12}+(v_3\sigma_{13})]+T(v_1\sigma_{21}+v_2\sigma_{22}+v_{23}\sigma_{23})$$

$$+A(v_1\sigma_{31}+v_2\sigma_{32}+v_2\sigma_{33})+v_1a_1+v_2a_2+v_3a_3+a_4+a_5+O\left(\frac{1}{n^2}\right) \qquad (7.1)$$

Evaluated at MLE = $(\hat{\alpha},\hat{\beta},\hat{\gamma})$ where;

$$a_1=\rho_1\sigma_{11}+\rho_2\sigma_{12}+\rho_3\sigma_{13} \qquad (7.2)$$

$$a_2=\rho_2\sigma_{21}+\rho_2\sigma_{22}+\rho_3\sigma_{23} \qquad (7.3)$$

$$a_3=\rho_1\sigma_{31}+\rho_2\sigma_{32}+\rho_3\sigma_{33} \qquad (7.4)$$

$$a_4=v_{12}\sigma_{12}+v_{13}\sigma_{13}+v_{23}\sigma_{23} \qquad (7.5)$$

$$a_5=\frac{1}{2}(v_{11}\sigma_{11}+v_{22}\sigma_{22}+v_{33}\sigma_{33}) \qquad (7.6)$$

$$S=[\sigma_{11}\ell_{111}+2\sigma_{12}\ell_{121}+2\sigma_{13}\ell_{131}+2\sigma_{23}\ell_{231}+\sigma_{22}\ell_{221}+\sigma_{33}\ell_{331}] \qquad (7.7)$$

$$T=[\sigma_{11}\ell_{112}+2\sigma_{12}\ell_{122}+2\sigma_{13}\ell_{132}+2\sigma_{23}\ell_{232}+\sigma_{22}\ell_{222}+\sigma_{33}\ell_{332}] \qquad (7.8)$$

$$A=[\sigma_{11}\ell_{113}+2\sigma_{12}\ell_{123}+2\sigma_{13}\ell_{133}+2\sigma_{23}\ell_{233}+\sigma_{22}\ell_{223}+\sigma_{33}\ell_{333}] \qquad (7.9)$$

For applying Lindley approximation to equation number (7.1)

$\Sigma_{ijk};\ I,j,k=1,2,3$

The Likelihood function described by equation number (3.1)

$\text{Log } R(t)=log\left(1-e^{-e^{-\beta(t-\gamma-\alpha)}}\right)$

$$\ell_1=\frac{\partial \log l}{\partial\alpha}=\frac{e^{-p}-\beta p}{1-e^{-p}}=-\beta p\frac{(1-R)}{R} \qquad (7.10)$$

$$\ell_2=\frac{\partial \log l}{\partial\beta}=\frac{e^{-p}}{1-e^{-p}}(-(t-\gamma-\alpha)p) \qquad (7.11)$$

$\ell_3 = \frac{\partial \log l}{\partial \gamma} = \frac{e^{-p} - \beta p}{1 - e^{-p}} = -\beta p \frac{(1-R)}{R}$ (7.12)

again

$\ell_{11} = \frac{\partial^2 \log l}{\partial \alpha^2} = \beta^2 \frac{1-R}{R}\left(p - \frac{p^2}{R}\right)$ (7.13)

$\ell_{22} = \frac{\partial^2 \log l}{\partial \beta^2} = (t - \gamma - \alpha)^2 \frac{1-R}{R}\left(p - \frac{p^2}{R}\right)$ (7.14)

$\ell_{33} = \frac{\partial^2 \log l}{\partial \gamma^2} = \beta^2 \frac{1-R}{R}\left(p - \frac{p^2}{R}\right)$ (7.15)

$\ell_{12} = \frac{\partial^2 \log l}{\partial \alpha\, \partial \beta} = (\beta(t - \gamma - \alpha) - 1)\frac{p(1-R)}{R} - \beta(t - \gamma - \alpha)\frac{p^2(1-R)}{R^2}$ (7.16)

$\ell_{21} = \frac{\partial^2 \log l}{\partial \beta\, \partial \alpha} = (\beta(t - \gamma - \alpha) - 1)\frac{p(1-R)}{R} - \beta(t - \gamma - \alpha)\frac{p^2(1-R)}{R^2}$ (7.17)

From (7.16) & (7.17)

$\ell_{12} = \ell_{21}$ (7.18)

$\ell_{13} = \frac{\partial^2 \log l}{\partial \alpha\, \partial \gamma} = \beta^2 \frac{1-R}{R}\left(p - \frac{p^2}{R}\right)$ (7.19)

$\ell_{31} = \frac{\partial^2 \log l}{\partial \gamma\, \partial \alpha} = \beta^2 \frac{1-R}{R}\left(p - \frac{p^2}{R}\right)$ (7.20)

From (7.19) & (7.20)

$\ell_{13} = \ell_{31}$ (7.21)

$\ell_{23} = \frac{\partial^2 \log l}{\partial \beta\, \partial \gamma} = \left(1 + \beta(t - \gamma - \alpha)\right)\frac{p(1-R)}{R} - \beta(t - \gamma - \alpha)\frac{p^2(1-R)}{R^2}$ (7.22)

$\ell_{32} = \frac{\partial^2 \log l}{\partial \gamma\, \partial \beta} = \left(1 + \beta(t - \gamma - \alpha)\right)\frac{p(1-R)}{R} - \beta(t - \gamma - \alpha)\frac{p^2(1-R)}{R^2}$ (7.23)

From (7.22) & (7.23)

$\ell_{23} = \ell_{32}$ (7.24)

Again,

$\ell_{111} = \frac{\partial^3 \log l}{\partial \alpha^3} = \beta^3 \frac{1-R}{R^2}(-p^3 + 3p^2 - pR)$ (7.25)

$\ell_{222} = \frac{\partial^3 \log l}{\partial \beta^3} = (t - \gamma - \alpha)^3 \frac{1-R}{R^2}(-p^3 + 3p^2 - pR)$ (7.26)

$\ell_{333} = \frac{\partial^3 \log l}{\partial \gamma^3} = \beta^3 \frac{1-R}{R^2}(-p^3 + 3p^2 - pR)$ (7.27)

$\forall, c = (t - \gamma - \alpha)$

$\ell_{112} = \frac{\partial}{\partial \alpha}\left[\frac{\partial^2 \log l}{\partial \alpha\, \partial \beta}\right] = 2\beta \frac{1-R}{R}\left(p - \frac{p^2}{R}\right) + \beta^2\left[-cp\frac{1-R}{R^2}\left(p - \frac{p^2}{R}\right) + c\frac{1-R}{R}\left(-p + \frac{2p^2R + p^3(1-R)}{R^2}\right)\right]$

(7.28) $\ell_{113} = \frac{\partial}{\partial \alpha}\left[\frac{\partial^2 \log l}{\partial \alpha\, \partial \gamma}\right] = \beta^3\left[\frac{1-R}{R^2}\left(p\left(p - \frac{p^2}{R}\right) - pR + 2p^2 + \frac{p^3(1-R)}{R}\right)\right]$ (7.29)

$\ell_{121} = \frac{\partial}{\partial \alpha}\left[\frac{\partial^2 \log l}{\partial \beta\, \partial \alpha}\right] = 2\beta \frac{1-R}{R^2}(-p^3 + 3p^2 - pR) + \beta^2\left[-cp\frac{1-R}{R^2}\left(p - \frac{p^2}{R}\right) + c\frac{1-R}{R}\left(-p + \frac{2p^2R + p^3(1-R)}{R^2}\right)\right]$

(7.30)

$\ell_{131} = \frac{\partial}{\partial \alpha}\left[\frac{\partial^2 \log l}{\partial \gamma\, \partial \alpha}\right] = \beta^2 \frac{1-R}{R^2}(-p^3 + 3p^2 - pR)$ (7.31)

$\ell_{221} = \frac{\partial}{\partial \beta}\left[\frac{\partial^2 \log l}{\partial \gamma\, \partial \alpha}\right] = -2c\frac{1-R}{R}\left(p - \frac{p^2}{R}\right) + c^2\frac{1-R}{R^2}(-p^3 + 3p^2 - pR)$ (7.32)

$\ell_{223} = \frac{\partial}{\partial \beta}\left[\frac{\partial^2 \log l}{\partial \beta\, \partial \gamma}\right] = -2c\frac{1-R}{R}\left(p - \frac{p^2}{R}\right) + c^2\left[-\beta p\frac{1-R}{R^2}\left(p - \frac{p^2}{R}\right) + \beta\frac{1-R}{R}\left(-p + \frac{2p^2R + p^3(1-R)}{R^2}\right)\right]$

(7.33)

$\ell_{232} = \frac{\partial}{\partial \beta}\left[\frac{\partial^2 \log l}{\partial \gamma\, \partial \beta}\right] = -2c\frac{1-R}{R}\left(p - \frac{p^2}{R}\right) + c^2\left[-\beta p\frac{1-R}{R^2}\left(p - \frac{p^2}{R}\right) + \beta\frac{1-R}{R}\left(-p + \frac{2p^2R + p^3(1-R)}{R^2}\right)\right]$

(7.34)

$\ell_{331} = \frac{\partial}{\partial \gamma}\left[\frac{\partial^2 \log l}{\partial \gamma\, \partial \alpha}\right] = \beta^2 \frac{1-R}{R^2}(-p^3 + 3p^2 - pR)$ (7.35)

$\ell_{332} = \frac{\partial}{\partial \gamma}\left[\frac{\partial^2 \log l}{\partial \gamma\, \partial \beta}\right] = 2\beta\frac{1-R}{R}\left(p - \frac{p^2}{R}\right) + \beta^2 c\left[-p\frac{1-R}{R^2}\left(p - \frac{p^2}{R}\right) + \frac{1-R}{R}\left(-p + \frac{2p^2R + p^3(1-R)}{R^2}\right)\right]$

(7.36)

$\ell_{231} = \frac{\partial}{\partial \beta}\left[\frac{\partial^2 \log l}{\partial \gamma\, \partial \alpha}\right] = -\beta\frac{p(1-R)}{R} + (1 + \beta c)\beta\left(\frac{p^2(1-R)}{R^2} - \frac{p(1-R)}{R}\right) + \beta\frac{p^2(1-R)}{R^2} + \beta^2 c\frac{1-R}{R^3}[2p^2R + p^3(R + 2(1-R))]$ (7.37)

$\ell_{122} = \frac{\partial}{\partial \alpha}\left[\frac{\partial^2 \log l}{\partial \beta^2}\right] = c\frac{p(1-R)}{R} - c(\beta c - 1)\left(\frac{p(1-R)}{R} + \frac{p^2(1-R)}{R^2}\right) - c\frac{p^2(1-R)}{R^2} + \beta c^2\frac{1-R}{R^3}[2p^2R + p^3(R + 2(1-R))]$ (7.38)

$\ell_{132} = \frac{\partial}{\partial \alpha}\left[\frac{\partial^2 \log l}{\partial \gamma\, \partial \beta}\right] = 2\beta\frac{1-R}{R}\left(p - \frac{p^2}{R}\right) + \beta^2 c\left[-p\frac{1-R}{R^2}\left(p - \frac{p^2}{R}\right) + \frac{1-R}{R}\left(-p + \frac{2p^2R + p^3(1-R)}{R^2}\right)\right]$

(7.39)

$$\ell_{133}=\frac{\partial}{\partial\alpha}\left[\frac{\partial^2\text{logl}}{\partial\gamma^2}\right]=\beta^2\left[-p\,\frac{1-R}{R^2}\left(p-\frac{p^2}{R}\right)+\frac{1-R}{R}\left(-p+\frac{2p^2R+p^3(1-R)}{R^2}\right)\right] \quad (7.40)$$

$$\ell_{233}=\frac{\partial}{\partial\beta}\left[\frac{\partial^2\text{logl}}{\partial\gamma^2}\right]=-\beta\,\frac{p(1-R)}{R}+(1+\beta c)\beta\left(\frac{p^2(1-R)}{R^2}-\frac{p(1-R)}{R}\right)+\beta\,\frac{p^2(1-R)}{R^2}+\beta^2c\,\frac{1-R}{R^3}\left[2p^2R+p^3\left(R+2(1-R)\right)\right] \quad (7.41)$$

$$\ell_{123}=\frac{\partial}{\partial\alpha}\left[\frac{\partial^2\text{logl}}{\partial\beta\,\partial\gamma}\right]=\beta^2c\,\frac{1-R}{R^3}\left(R\left((-p^3+3p^2-pR)\right)+2p^3\right) \quad (7.42)$$

Now, From equation number (7.25) to (7.42)

$$-\left[\ell_{\text{ijk}}\right]=\begin{bmatrix}\ell_{111}\ell_{112}\ell_{113}\\ \ell_{221}\ell_{222}\ell_{223}\\ \ell_{331}\ell_{332}\ell_{333}\end{bmatrix}$$

$$-\left[\ell_{\text{ijk}}\right]=$$

$$\begin{bmatrix}\beta^3\frac{1-R}{R^2}(-p^3+3p^2-pR) & 2\beta\frac{1-R}{R}\left(p-\frac{p^2}{R}\right)+\beta^2\begin{bmatrix}-cp\frac{1-R}{R^2}\left(p-\frac{p^2}{R}\right)+\\ c\frac{1-R}{R}\left(-p+\frac{2p^2R+p^3(1-R)}{R^2}\right)\end{bmatrix} & \beta^3\left[\frac{1-R}{R^2}\left(p\left(p-\frac{p^2}{R}\right)-pR+2p^2+\frac{p^3(1-R)}{R}\right)\right]\\ -2c\frac{1-R}{R}\left(p-\frac{p^2}{R}\right)+c^2\frac{1-R}{R^2}\begin{pmatrix}-p^3\\+3p^2-pR\end{pmatrix} & (t-\gamma-\alpha)^3\frac{1-R}{R^2}(-p^3+3p^2-pR) & -\beta\frac{p(1-R)}{R}+(1+\beta c)\beta\begin{pmatrix}\frac{p^2(1-R)}{R^2}\\-\frac{p(1-R)}{R}\end{pmatrix}+\beta\frac{p^2(1-R)}{R^2}+\beta^2c\frac{1-R}{R^3}\begin{bmatrix}2p^2R+\\p^3(R+\\2(1-R))\end{bmatrix}\\ \beta^2\frac{1-R}{R^2}(-p^3+3p^2-pR) & 2\beta\frac{1-R}{R}\left(p-\frac{p^2}{R}\right)+\beta^2c\begin{bmatrix}-p\frac{1-R}{R^2}\left(p-\frac{p^2}{R}\right)\\+\frac{1-R}{R}\left(-p+\frac{2p^2R+p^3(1-R)}{R^2}\right)\end{bmatrix} & \beta^3\frac{1-R}{R^2}(-p^3+3p^2-pR)\end{bmatrix} \quad (7.43)$$

$$=\begin{bmatrix}K_{11}K_{12}K_{13}\\ K_{21}K_{22}K_{23}\\ K_{31}K_{32}K_{33}\end{bmatrix}$$

Determinantal value of $-\left[\ell_{\text{ijk}}\right]$

$$D=-\begin{Bmatrix}K_{11}(K_{22}K_{33}-K_{23}K_{32})+K_{12}(K_{21}K_{33}-K_{23}K_{31})+\\ (K_{21}K_{32}-K_{31}K_{22})\end{Bmatrix}$$

Adjoint of matrix = - $\left\lfloor \ell_{\text{ijk}} \right\rfloor$

Cofactor of matrix = - $\left\lfloor \ell_{\text{ijk}} \right\rfloor$

$$a_{11}=[K_{22}K_{33}-K_{23}K_{32}]=M_{11}$$
$$a_{12}=-[K_{21}K_{33}-K_{23}K_{31}]$$
$$=K_{23}K_{31}-K_{21}K_{33}=M_{12}$$
$$a_{13}=[K_{21}K_{32}-K_{22}K_{31}]=M_{13}$$
$$a_{21}=[K_{12}K_{33}-K_{32}K_{13}]$$
$$=K_{32}K_{13}-K_{12}K_{33}=M_{21}$$
$$a_{22}=[K_{12}K_{33}-K_{31}K_{13}]=M_{22}$$
$$a_{23}=[K_{11}K_{32}-K_{12}K_{31}]$$
$$=K_{12}K_{31}-K_{11}K_{32}=M_{23}$$
$$a_{31}=[K_{12}K_{23}-K_{13}K_{22}]=M_{31}$$
$$a_{32}=-[K_{11}K_{23}-K_{13}K_{21}]$$
$$=[K_{13}K_{21}-K_{11}K_{23}]=M_{32}$$
$$a_{33}=[K_{12}K_{22}-K_{12}K_{21}]=M_{33}$$

$$\left[-\ell_{\text{ijk}}\right]^{-1}=\frac{\text{Adjoint of}\left[\ell_{\text{ijk}}\right]}{\left|-\ell_{\text{ijk}}\right|}=\begin{bmatrix}\frac{M_{11}}{D}\frac{M_{21}}{D}\frac{M_{13}}{D}\\ \frac{M_{12}}{D}\frac{M_{22}}{D}\frac{M_{23}}{D}\\ \frac{M_{13}}{D}\frac{M_{23}}{D}\frac{M_{33}}{D}\end{bmatrix}$$

$$=\begin{bmatrix}\sigma_{11}\sigma_{21}\sigma_{13}\\ \sigma_{12}\sigma_{22}\sigma_{23}\\ \sigma_{13}\sigma_{23}\sigma_{33}\end{bmatrix}$$

Approximate Bayes Estimator Approach

$$V(\alpha,\beta,\gamma)=v$$
$$\hat{V}_{AB}=E\left(v|\underline{x}\right)$$

evaluated from equation number (6.4) and (7.1) joint prior density

$$J(\alpha,\beta,\gamma)=j_1(\alpha)j_2(\gamma)j_3\left(\frac{\beta}{\gamma}\right)$$

$$=\frac{c}{\partial\Gamma\xi}\gamma^{-\xi}\beta^{\xi+1}\exp\left[-\left(\frac{\gamma}{\partial}+\frac{\beta}{\gamma}\right)\right]$$

$$\rho=\log J=\log c-\log\partial-\log\Gamma\xi\,(\xi+1)\log\beta-\xi\log\gamma-\left(\frac{\gamma}{\delta}+\frac{\beta}{\gamma}\right) \quad (8.1)$$

$$\rho_1=\frac{\partial\rho}{\partial\alpha}=0 \quad (8.2)$$

$$\rho_2=\frac{\partial\rho}{\partial\beta}=\frac{\xi+1}{\beta}\frac{-1}{\gamma} \quad (8.3)$$

$$\rho_3=\frac{\partial\rho}{\partial\gamma}=\frac{-\xi}{\gamma}\frac{-1}{\delta}+\frac{1}{\gamma^2} \quad (8.4)$$

By changing S, T, and A in equations (7.43) and (7.28) to (7.42)
we obtain,

$$S=[\sigma_{11}\ell_{111}+2\sigma_{12}\ell_{121}+2\sigma_{13}\ell_{131}+2\sigma_{23}\ell_{231}+\sigma_{22}\ell_{221}+\sigma_{33}\ell_{331}]$$

$$=\sigma_{11}\beta^3\frac{1-R}{R^2}(-p^3+3p^2-pR)+\ 2\sigma_{12}(-2c\frac{1-R}{R}\left(p-\frac{p^2}{R}\right)+c^2\frac{1-R}{R^2}(-p^3+3p^2-pR))+$$
$$2\sigma_{13}\beta^2\frac{1-R}{R^2}(-p^3+3p^2-pR)\ +2\sigma_{23}\left(-\beta\frac{p(1-R)}{R}+(1+\beta c)\beta\left(\frac{p^2(1-R)}{R^2}-\frac{p(1-R)}{R}\right)+\beta\frac{p^2(1-R)}{R^2}+\right.$$
$$\left.\beta^2c\frac{1-R}{R^3}[2p^2R+p^3(R+2(1-R))]]\right)+\sigma_{22}\left(-2c\frac{1-R}{R}\left(p-\frac{p^2}{R}\right)+c^2\frac{1-R}{R^2}(-p^3+3p^2-pR)\right)\ +$$
$$\sigma_{33}\beta^2\frac{1-R}{R^2}(-p^3+3p^2-pR) \quad (8.5)$$

$$T=[\sigma_{11}\ell_{112}+2\sigma_{12}\ell_{122}+2\sigma_{13}\ell_{132}+2\sigma_{23}\ell_{232}+\sigma_{22}\ell_{222}+\sigma_{33}\ell_{332}]$$

$$=\sigma_{11}2\beta\frac{1-R}{R}\left(p-\frac{p^2}{R}\right)+\beta^2\left[-cp\frac{1-R}{R^2}\left(p-\frac{p^2}{R}\right)+c\frac{1-R}{R}\left(-p+\frac{2p^2R+p^3(1-R)}{R^2}\right)\right]+2\sigma_{12}c\frac{p(1-R)}{R}-$$
$$c(\beta c-1)\left(\frac{p(1-R)}{R}+\frac{p^2(1-R)}{R^2}\right)-c\frac{p^2(1-R)}{R^2}+\beta c^2\frac{1-R}{R^3}[2p^2R+p^3(R+2(1-R))]+2\sigma_{13}2\beta\frac{1-R}{R}\left(p-\frac{p^2}{R}\right)+$$
$$\beta^2c\left[-p\frac{1-R}{R^2}\left(p-\frac{p^2}{R}\right)+\frac{1-R}{R}\left(-p+\frac{2p^2R+p^3(1-R)}{R^2}\right)\right]+2\sigma_{23}\left(-2c\frac{1-R}{R}\left(p-\frac{p^2}{R}\right)+c^2\left[-\beta p\frac{1-R}{R^2}\left(p-\frac{p^2}{R}\right)+\right.\right.$$
$$\left.\left.\beta\frac{1-R}{R}\left(-p+\frac{2p^2R+p^3(1-R)}{R^2}\right)\right]\right)+\sigma_{22}(t-\gamma-\alpha)^3\frac{1-R}{R^2}(-p^3+3p^2-pR)+\sigma_{33}2\beta\frac{1-R}{R}\left(p-\frac{p^2}{R}\right)+$$
$$\beta^2c\left[-p\frac{1-R}{R^2}\left(p-\frac{p^2}{R}\right)+\frac{1-R}{R}\left(-p+\frac{2p^2R+p^3(1-R)}{R^2}\right)\right] \quad (8.6)$$

$$A=[\sigma_{11}\ell_{113}+2\sigma_{12}\ell_{123}+2\sigma_{13}\ell_{133}+2\sigma_{23}\ell_{233}+\sigma_{22}\ell_{223}+\sigma_{33}\ell_{333}]$$

$$=\sigma_{11}\beta^3\left[\frac{1-R}{R^2}\left(p\left(p-\frac{p^2}{R}\right)-pR+2p^2+\frac{p^3(1-R)}{R}\right)\right]+2\sigma_{12}\beta^2c\frac{1-R}{R^3}\left(R\left((-p^3+3p^2-pR)\right)+2p^3\right)+$$
$$2\sigma_{13}\beta^2\left[-p\frac{1-R}{R^2}\left(p-\frac{p^2}{R}\right)+\frac{1-R}{R}\left(-p+\frac{2p^2R+p^3(1-R)}{R^2}\right)\right]+2\sigma_{23}\left(-\beta\frac{p(1-R)}{R}+(1+\beta c)\beta\left(\frac{p^2(1-R)}{R^2}-\frac{p(1-R)}{R}\right)+\right.$$
$$\left.\beta\frac{p^2(1-R)}{R^2}+\beta^2c\frac{1-R}{R^3}[2p^2R+p^3(R+2(1-R))]]\right)\sigma_{22}\left(-2c\frac{1-R}{R}\left(p-\frac{p^2}{R}\right)+c^2\left[-\beta p\frac{1-R}{R^2}\left(p-\frac{p^2}{R}\right)+\right.\right.$$
$$\left.\left.\beta\frac{1-R}{R}\left(-p+\frac{2p^2R+p^3(1-R)}{R^2}\right)\right]\right)+\sigma_{33}\beta^3\frac{1-R}{R^2}(-p^3+3p^2-pR) \quad (8.7)$$

$$\widehat{V}_{AB}=E(v|\underline{x})=v+(v_1\alpha_1+v_2\alpha_2+v_3\alpha_3+\alpha_4+\alpha_5)+\frac{1}{2}[(S\sigma_{11}+\ T\sigma_{21}+A\sigma_{31})v_1+(S\sigma_{12}+T\sigma_{22}+$$
$$A\sigma_{32})v_2+(S\sigma_{13}+T\sigma_{23}+A\sigma_{33})v_3]$$

$$=v+\phi_1+\phi_2 \quad (8.8)$$

Were

$$\varphi_1=(v_1a_1+v_2a_2+v_3a_3+a_4+a_5) \quad (8.9)$$

$$\varphi_2=\frac{1}{2}[(S\sigma_{11}+T\sigma_{21}+A\sigma_{31})v_1+(S\sigma_{12}+T\sigma_{22}+A\sigma_{32})v_2$$
$$+(S\sigma_{13}+T\sigma_{23}+A\sigma_{33})v_3] \quad (8.10)$$

evaluated at the MLE $\widehat{V}\ =(\widehat{\alpha},\widehat{\beta},\widehat{\gamma})$) where;

$$a_1=\rho_1\sigma_{11}+\rho_2\sigma_{12}+\rho_3\sigma_{13}$$
$$=o.\sigma_{11}+\left[\frac{\xi+1}{\beta}-\frac{1}{\gamma}\right]\sigma_{12}+\left[\frac{-\xi}{\gamma}-\frac{1}{\delta}+\frac{1}{\gamma^2}\right]\sigma_{13} \quad (8.11)$$

$$a_2=\rho_1\sigma_{21}+\rho_2\sigma_{22}+\rho_3\sigma_{23}$$
$$=\left[\frac{\xi+1}{\beta}-\frac{1}{\gamma}\right]\sigma_{22}+\left[\frac{-\xi}{\gamma}-\frac{1}{\delta}+\frac{1}{\gamma^2}\right]\sigma_{23} \quad (8.12)$$

$$a_3=\rho_1\sigma_{31}+\rho_2\sigma_{32}+\rho_3\sigma_{33}$$
$$=\left[\frac{\xi+1}{\beta}-\frac{1}{\gamma}\right]\sigma_{32}+\left[\frac{-\xi}{\gamma}-\frac{1}{\delta}+\frac{1}{\gamma^2}\right]\sigma_{33} \quad (8.13)$$

$$a_4=v_{12}\sigma_{12}+v_{13}\sigma_{13}+v_{23}\sigma_{23} \quad (8.14)$$

$a_5 = \frac{1}{2}(v_{11}\sigma_{11} + v_{22}\sigma_{22} + v_{33}\sigma_{33})$ (8.15)

Bayesian Approximation of Reliability Gumbel distribution function with three parameters under the precautionary loss function.

$\widehat{R}_{ABP} = [E_h(R^2)]^{\frac{1}{2}}$ (9.1)

$\widehat{R}_{ABP} = \frac{\iiint R^2 J^*(\alpha,\beta,\gamma)d\alpha d\beta d\gamma}{\iiint J^*(\alpha,\beta,\gamma)d\alpha d\beta d\gamma}$ (9.2)

The Lindely Approximation technique is applied to simplify the equation

$\widehat{R}_{ABP} = E[v|\underline{x}]$

$= v^2 + \varphi_1 + \varphi_2$ (9.3)

$V = R^2 = \left[1 - e^{-e^{-\beta(t-\gamma-\alpha)}}\right]^2$

$\log V = 2\log\left(1 - e^{-e^{-\beta(t-\gamma-\alpha)}}\right)$

$\frac{1}{V}\frac{dV}{d\alpha} = 2\beta\frac{e^{-e^{-\beta(t-\gamma-\alpha)}}e^{-e^{-\beta(t-\gamma-\alpha)}}}{1 - e^{-e^{-\beta(t-\gamma-\alpha)}}} = 2\beta\frac{p(1-R)}{R}$ (9.4)

$\forall, c = (t-\gamma-\alpha), p = e^{-\beta(t-\gamma-\alpha)} and\ R = 1 - e^{-e^{-\beta(t-\gamma-\alpha)}}$

$V_1 = 2\beta p(1-R)R$ (9.5)

$V_{11} = 2\beta^2[p(1-R)R(1-p) + p^2(1-R)^2]$ (9.6)

$V_{12} = 2p(1-R)R - 2\beta cp(1-R)[R(1-p) + p(1-R)]$ (9.7)

$V_{13} = 2\beta^2[p(1-R)R(1-p) + p^2(1-R)^2]$ (9.8)

$V_2 = -2cp(1-R)R$ (9.9)

$V_{22} = 2c^2[p(1-R)R(1-p) + p^2(1-R)^2]$ (9.10)

$V_{23} = 2p(1-R)R - 2\beta cp(1-R)[R(1-p) + p(1-R)]$ (9.11)

$V_3 = 2\beta p(1-R)R$ (9.12)

$V_{31} = 2\beta^2[p(1-R)R(1-p) + p^2(1-R)^2]$ (9.13)

$V_{32} = 2p(1-R)R - 2\beta cp(1-R)[R(1-p) + p(1-R)]$ (9.14)

$V_{33} = 2\beta^2[p(1-R)R(1-p) + p^2(1-R)^2]$ (9.15)

$\widehat{R}_{ABP} = E[R^2|\underline{x}] = R^2 + \varphi_1 + \varphi_2$

$\varphi_1 = (v_1a_1 + v_2a_2 + v_3a_3 + a_4 + a_5)$

$\varphi_2 = \frac{1}{2}[(S\sigma_{11} + T\sigma_{21} + A\sigma_{31})v_1 + (S\sigma_{12} + T\sigma_{22} + A\sigma_{32})v_2 + (S\sigma_{13} + T\sigma_{23} + A\sigma_{33})v_3]$

$\varphi_1 = 2\beta p(1-R)Ra_1 - 2cp(1-R)Ra_2 + 2\beta p(1-R)Ra_3 + 2p(1-R)R + 2p(1-R)R - 2\beta cp(1-R)[R(1-p) + p(1-R)]\sigma_{12} + 2\beta^2[p(1-R)R(1-p) + p^2(1-R)^2]\sigma_{13} + 2p(1-R)R - 2\beta cp(1-R)[R(1-p) + p(1-R)]\sigma_{23} + \frac{1}{2}(2\beta^2[p(1-R)R(1-p) + p^2(1-R)^2]\sigma_{11} + 2c^2[p(1-R)R(1-p) + p^2(1-R)^2]\sigma_{22} + 2\beta^2[p(1-R)R(1-p) + p^2(1-R)^2]\sigma_{33})$

$\varphi_1 = 2p(1-R)\{R(\beta a_1 - ca_2 + \beta a_3) + 3R - \beta c[R(1-p) + p(1-R)](\sigma 12 + \sigma 23) + \beta^2[p(1-R)R(1-p)](\sigma_{11} + \sigma_{13} + \sigma_{33}) + 2c^2[p(1-R) + R(1-p)]\sigma_{22}\}$ (9.16)

$\forall, \Delta_1 = \{R(\beta a_1 - ca_2 + \beta a_3) + 3R - \beta c[R(1-p) + p(1-R)](\sigma 12 + \sigma 23) + \beta^2[p(1-R)R(1-p)](\sigma_{11} + \sigma_{13} + \sigma_{33}) + 2c^2[p(1-R) + R(1-p)]\sigma_{22}\}$ (9.17)

$\varphi_1 = 2p(1-R)\Delta_1$ (9.18)

$$\varphi_2 = \frac{1}{2}[(S\sigma_{11} + T\sigma_{21} + A\sigma_{31})2\beta p(1-R)R + (S\sigma_{12} + T\sigma_{22} + A\sigma_{32}) - 2cp(1-R)R + (S\sigma_{13} + T\sigma_{23} + A\sigma_{33})2\beta p(1-R)R] \quad (9.19)$$

$\varphi_2 = p(1-R)R[\beta(S(\sigma_{11} + \sigma_{13}) + T(\sigma_{21} + \sigma_{23}) + A(\sigma_{31} + \sigma_{33})) + A\sigma_{32}] - c(S\sigma_{12} + T\sigma_{22} + A\sigma_{32})$
(9.20)

$\forall, \Delta_2 = \quad [\beta(S(\sigma_{11} + \sigma_{13}) + T(\sigma_{21} + \sigma_{23}) + A(\sigma_{31} + \sigma_{33})) + A\sigma_{32}] - c(S\sigma_{12} + T\sigma_{22} + A\sigma_{32})$
(9.21)

$\varphi_2 = p(1-R)R\Delta_2$ (9.22)

$E[R^2|\underline{x}] = R^2 + \varphi_1 + \varphi_2$

$= R^2 + 2p(1-R)\Delta_1 + p(1-R)R\Delta_2$

$= R^2 + p(1-R)[2\Delta_1 + R\Delta_2]$

$\widehat{R}_{ABP} = [(R^2 + p(1-R)[2\Delta_1 + R\Delta_2])]^{\frac{1}{2}}$ (9.23)

2.3. Simulation Design

The present analysis focuses on evaluating the parameters α, β, and γ, which represent the shape, scale, and location aspects of the three-parameter Gumbel distribution. The true values of these parameters are fixed as α = 1, β = 0.5, and γ = 1.5. Two estimation methods are compared in this study: Under the precautionary loss function (PLF), Lindley's approximation is used for both Maximum Likelihood Estimation (MLE) and Approximate Bayesian Estimation (ABS). Hyperparameters are used to provide the prior distributions for the Bayesian technique, with ξ = 5 for β and δ = 1.5 for γ. Using data from the Gumbel distribution, 500 replications of the simulation experiment are carried out for a range of sample sizes, from 10 to 80, in increments of 10. MLE and ABS estimates for α, β, and γ are generated in each iteration after a dataset is simulated using the fixed true parameter values. The mean and mean squared error (MSE) of the parameter estimations are computed to summarise the findings for each sample size. The MLE and ABS estimates, together with their corresponding MSEs (given in square brackets), are displayed in each column of the table that presents these data. A distinct sample size is shown by each row in the table. This study's primary objective is to compare the accuracy and stability of the MLE and ABS approaches. With an emphasis on minimising variations in the outcomes across various sample sizes, the comparison aids in assessing how well each approach approximates the genuine parameter values.

Table 1: MSEs and mean estimates for α, β, and γ under ABS and MLE

n	$\widehat{\alpha}_{MLE}$	$\widehat{\alpha}_{ABS}$	$\widehat{\beta}_{MLE}$	$\widehat{\beta}_{ABS}$	$\widehat{\gamma}_{MLE}$	$\widehat{\gamma}_{ABS}$
10	1.040877 [0.1171523464]]	1. 171317 [0. 1342616048]	0.576938 [0.0343091354]	1.113162 [0.4840218382]	1.540877 [0.1171523464]	0.948648 [0.5067567336]
20	1.024228 [0.0583357218]	1.013030 [0.0018260628]	0.542795 [0.0126940656]	0.836768 [0.1389459041]	1.524228 [0.0583357218]	0.963034 [0.3973299360]
30	1.013747 [0.0372447231]	1.061572 [0.0396491375]	0.524237 [0.007572130]	0.727362 [0.0648065565]	1.513747 [0.0372447231]	1.149942 [0.1752145879]
40	1.004663 [0.0240499528]	1.040188 [0.0247844456]	0.523854 [0.0051292999]	0.682058 [0.0409712245]	1.504663 [0.0240499528]	1.250360 [0.0917213504]
50	1.015312 [0.0232263927]	1.044064 [0.0243037624]	0.516628 [0.0035442993]	0.645868 [0.0263180054]	1.515312 [0.0232263927]	1.313480 [0.0598147024]
60	1.008723 [0.0175004583]	1.033152 [0.0181406353]	0.511790 [0.0028988590]	0.620107 [0.0184945857]	1.508723 [0.0175004583]	1.339902 [0.0440572339]
70	1.011172 [0.0167835645]	1.032261 [0.0173550794]	0.508466 [0.0024167543]	0.601678 [0.0136766058]	1.511172 [0.0167835645]	1.366856 [0.0350944522]
80	1.006378 [0.0127294611]	1.024722 [0.0130776072]	0.509982 [0.0021898178]	0.591875 [0.0111771135]	1.506378 [0.0127294611]	1.382013 [0.0270584167]

III. Analysis of Estimation under the Precautionary Loss Function

Two techniques were used to estimate the parameters for the three-parameter Gumbel distribution: Maximum Likelihood Estimation (MLE) and Approximate Bayesian Estimation (ABS). The ABS was computed using Lindley's approximation under the cautious loss function. The objective was to estimate the distribution's three parameters: the location parameter (γ), scale parameter (β), and shape parameter (α).

The research was conducted for a range of sample sizes, from n = 10 to n = 80. Fixed true values of the parameters were used to create 500 datasets for each sample size. Each dataset was subjected to both the MLE and ABS techniques, and the accuracy was determined by calculating the average estimates and the Mean Squared Error (MSE). The table displays the MSE values in square brackets.

To better understand how the estimators perform, smoothed graphs were created for each parameter. These graphs show how the estimates move closer to the true values as the sample size increases. This helps to clearly see which method performs better, especially when the data size is small.

Overall, this analysis helps to compare the classical MLE approach with the Bayesian ABS method under the precautionary loss function and shows how reliable each method is for different sample sizes.

Fig. 1

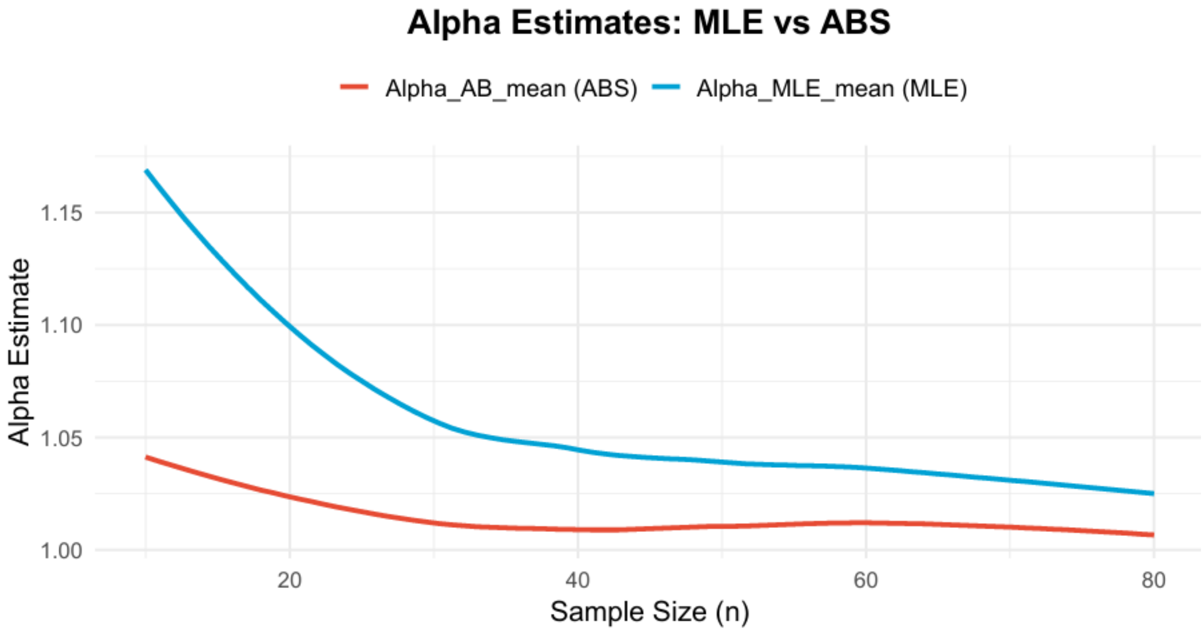
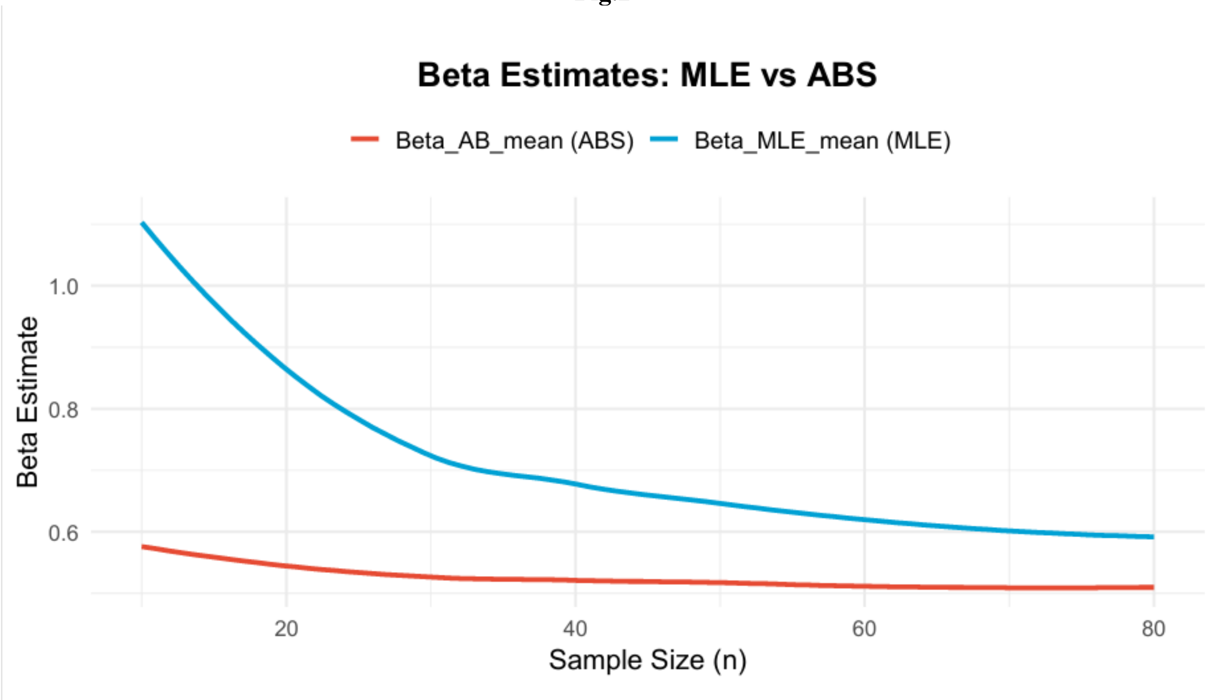


Figure 1 provides a detailed comparison of how the estimates for the shape parameter (α) behave across different sample sizes. The shape parameter plays a critical role in determining the form and skewness of the three-parameter Gumbel distribution, making its accurate estimation very important. When the sample size is small ($n = 10$ or 20), the Maximum Likelihood Estimator (MLE) start higher (around $\alpha \approx 1.17$) than the true value of $\alpha = 1$, indicating a minor upward bias. This occurs because MLE depends entirely on the sample data, and with limited data, the influence of random variation causes minor overvaluation and bias.

On the other hand, the Approximate Bayesian Estimator (ABS), which incorporates prior information through the precautionary loss function, remains much closer to the true value even with smaller datasets. The ABS estimates are smoother and more stable, showing less fluctuation compared to MLE. This is because the Bayesian approach balances the information coming from both the data and the prior, thereby reducing the uncertainty that arises when data are limited.

As the sample size increases ($n \geq 40$), both estimation methods begin to improve and converge toward the true value of α. However, even at larger sample sizes, ABS consistently shows slightly lower Mean Squared Error (MSE) and less variability than MLE, indicating that it maintains a small but clear advantage. The smoother trend of the ABS estimates in the graph reflects its reliability and efficiency, while the rough fluctuations of the MLE line, especially at lower n, highlight its sensitivity to random sample variation.

Fig.2



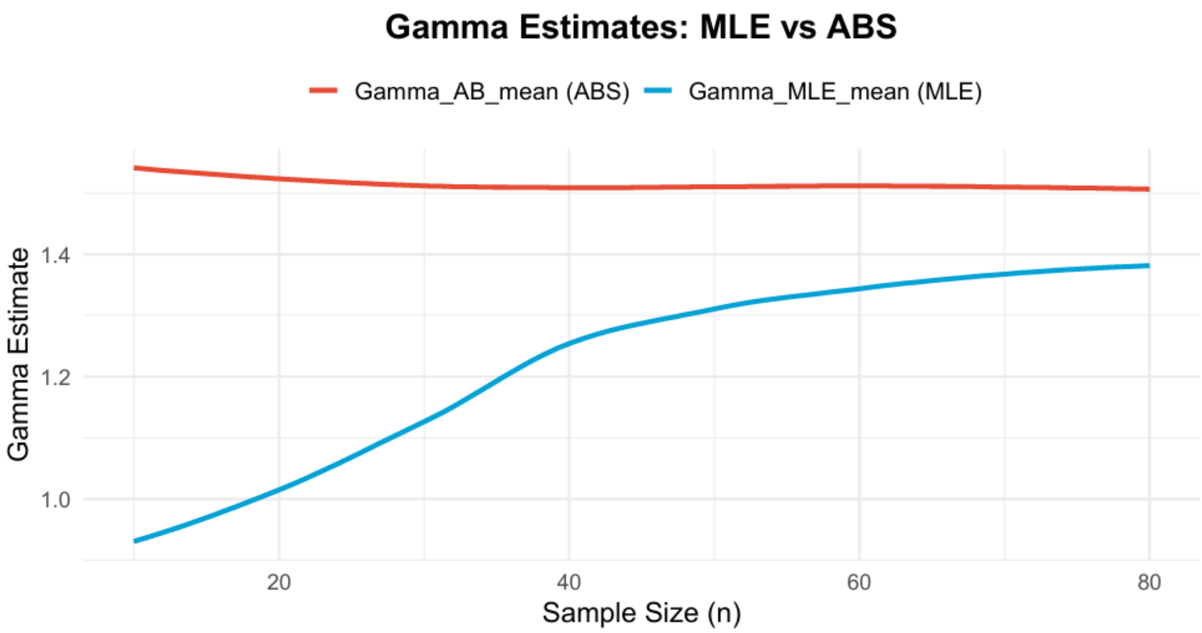
The behaviour of the estimates for the scale parameter (β) over various sample sizes is shown in detail in Figure 2. The spread or variability of the three-parameter Gumbel distribution is mostly determined by the scale parameter. Since β directly influences the breadth of the distribution and, in turn, the dependability of modelling and prediction, accurate estimation of β is crucial.

When the sample size is small (n = 10 or 20), the Maximum Likelihood Estimator (MLE) tends to overestimate β, meaning it provides values higher than the true value of β = 0.5. This overestimation occurs because MLE is highly sensitive to random variation in small datasets. When the sample size is small, there are fewer data points available, which makes the estimation process more sensitive to unusual or extreme observations. These extreme values can have a strong impact on the likelihood function, causing the estimates to shift away from the true value. This results in biased and unstable estimates. In the figure, this effect is seen as the MLE curve starting above the true parameter line, indicating a consistent tendency to overestimate the value of β.

On the other hand, the Approximate Bayesian Estimator (ABS), which uses prior information and the precautionary loss function, produces estimates that remain much closer to the true value of β, even when the sample size is very small. The Bayesian approach effectively combines the limited sample information with prior knowledge about the parameter, which acts as a stabilizing factor. This prevents the estimates from conflicting too far from the true value and minimizes the bias caused by small sample fluctuations. As a result, the ABS estimates are smoother and show a gradual, consistent trend toward the true β=0.5 value.

As the sample size increases (n ≥ 40), both estimation methods improve significantly. The additional data reduce the impact of random variation, leading to more accurate and stable estimates for both MLE and ABS. At larger sample sizes (n = 70, 80), the two methods nearly converge and give estimates that are very close to the true value of β=0.5. However, even in this scenario, ABS maintains a slight advantage, as reflected in its lower Mean Squared Error (MSE) and smoother estimation curve. The smoothness of the ABS line indicates that the estimates are less sensitive to random noise, while the MLE curve shows small fluctuations, particularly in mid-range sample sizes.

Fig.3



A thorough comparison of the estimation of the location parameter (γ) under the cautious loss function for various sample sizes using the Maximum Likelihood Estimator (MLE) and the Approximate Bayes Estimator (ABS) is shown in Figure 3. Because it establishes the starting point of the distribution and effectively shifts the entire distribution along the horizontal axis, the location parameter is crucial. Precise estimation of γ is crucial since even little mistakes can cause the entire distribution to change, which could affect how the data is interpreted and how the other parameters are estimated.

In comparison to their performance for the shape (α) and scale (β) parameters, both MLE and ABS perform rather well when the sample size is relatively small (n = 10 or 20). The estimates of γ remain quite near to its real value of γ = 1.5 from the outset, indicating that this parameter is naturally simpler to estimate. Nonetheless, a little distinction may be shown even in this small-sample situation:

The MLE estimates exhibit slightly more fluctuation below the true value due to their complete dependence on the limited sample data.

The ABS estimates, in contrast, are smoother and more consistent because the Bayesian approach uses prior information to stabilize the estimates, keeping them close to the true value when the dataset is small.

As the sample size increases (n ≥ 40), both methods rapidly improve, and their estimates converge almost perfectly to the true value of γ. At this point, the differences between MLE and ABS become minimal. However, ABS still shows a slight advantage, as reflected in its lower Mean Squared Error (MSE), smoother convergence pattern.

The figure clearly shows that the curves for both MLE and ABS stay very close to the true parameter line throughout the range of sample sizes. The ABS curve remains nearly flat and smooth. While the MLE curve has small upward fluctuations, especially in the mid-range sample sizes (n = 30–50). This smoothness indicates that ABS produces more reliable results by reducing random variability, even when the sample size is not very large.

The overall analysis confirms that the proposed ABS method consistently achieves lower Mean Squared Error (MSE) values than the classical MLE for all parameters and sample sizes. Similar findings were observed in earlier studies by Srivastava & Yadav (2018), Jabarali et al. (2024), and Okumu et al. (2024), who also reported improved stability through Bayesian estimation. However, none of these studies incorporated the precautionary loss framework for the three-parameter Gumbel model. The present results therefore establish a distinct advancement by combining Lindley's approximation with an asymmetric loss structure, providing more reliable estimation for small-sample reliability data.

IV. Conclusion

In order to estimate the three parameters (α, β, and γ) of the three-parameter Gumbel distribution, this work devised an Approximate Bayesian Estimation (ABS) approach employing Lindley's approximation under the cautious loss function. ABS consistently provided lower Mean Squared Errors and more stable estimates than the conventional Maximum Likelihood Estimation (MLE) approach, according to simulation trials conducted over a range of sample sizes (10–80).

The main advantage of the suggested method is that it may produce accurate and trustworthy findings even for tiny or extreme-value datasets, where MLE is prone to bias. The model can more accurately represent real-world dependability scenarios where underestimating risk is more expensive than overestimating it thanks to the addition of an asymmetric loss structure.

While computational effort slightly increases due to the Bayesian integration, the accuracy gain balances this limitation. Future work may extend this framework to censored reliability data or other extreme-value models to further test its robustness. Overall, the proposed ABS approach provides a novel, efficient, and realistic tool for reliability estimation in engineering and applied sciences.

References

[1]. Sinha SK. Reliability and life testing. New Delhi: Wiley Eastern; 1986.

[2]. Lindley DV. Approximate Bayesian methods. *Trabajos de Estadística y de Investigación Operativa*. 1980;31(1):223–245. doi:10.1007/BF02888369

[3]. Barlow RE, Proschan F. *Mathematical Theory of Reliability*. Philadelphia: SIAM; 1996.

[4]. Kotz S, Nadarajah S. *Extreme Value Distributions: Theory and Applications*. London: Imperial College Press; 2000.

[5]. Coles S. *An Introduction to Statistical Modelling of Extreme Values*. London: Springer; 2001.

[6]. O'Connor PDT, Kleyner A. *Practical Reliability Engineering*. 5th ed. Hoboken (NJ): Wiley; 2012.

[7]. Elsayed EA. *Reliability Engineering*. 2nd ed. Hoboken (NJ): Wiley; 2012.

[8]. Bernardo JM, Smith AFM. *Bayesian Theory*. Chichester: Wiley; 2009.

[9]. Modarres M, Kaminskiy M, Krivtsov V. *Reliability Engineering and Risk Analysis: A Practical Guide*. 3rd ed. Boca Raton (FL): CRC Press; 2016.

[10]. Srivastava U, Yadav P. Approximate Bayes estimate for the parameters of generalized compound Rayleigh distribution under precautionary loss function. *Int J Creat Res Thoughts*. 2018;6(2):192–198.

[11]. Okumu OK, Ouno OJ, Karanjah AN, Muthiga SN. Three-parameter Gumbel distribution: formulation and parameter estimation. *Int J Stat Distrib Appl*. 2024;10(2):25–37. doi:10.11648/j.ijsd.20241002.12

[12]. Pandey J, Srivastava U. Approximating the three parameters of Gumbel model through Lindley Bayes. *Int J Stat Appl Math*. 2025;10(8):213–225. doi:10.22271/maths.2025.v10.i8c.2144

[13]. Jabarali N, Rajasekaran V, Khalid M. Bayesian inference for lifetime models under asymmetric loss. *Mathematics*. 2024;12(2):254. doi:10.3390/math12020254

[14]. Singh A, Kumar P. Bayesian estimation in reliability analysis under generalized loss functions. *Reliab Eng Syst Saf*. 2023;234:109063. doi:10.1016/j.ress.2023.109063