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Linearization Of The Nonlinear Heat Equation

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Abstract:
In this article the problem of solving the nonlinear heat equation is raised, this is one of many of which a solution can be found by making a change of variable, or by making a Taylor development around a point to work on, in our In the case it is a transport phenomenon whose equation is non-linear, the reason for being non-linear corresponds to the thermal conductivity of the material, which is a function of the temperature, which is what is generally found in all real cases, since few materials are one hundred percent pure. For this type of case in particular, a method is used which consists of modifying the thermal conductivity function using the Gustav Robert Kirchhoff transform, which has the purpose of reducing the nonlinear equation to a linear one.

***Background**: In 1859, Gustav Robert Kirchhoff (1824-1887) obtained, from the second law of Thermodynamics, that objects cannot be differentiated by their thermal radiation at a given temperature. In 1860 Kirchhoff established the definition of the black body as capable of absorbing all the incident radiation, he even modeled it as a chamber with a small hole for the radiation to enter.*

***Materials and Methods**: The Gustav Robert Kirchhoff transformation is used to linearize the nonlinear equation, this nonlinearity occurs when the thermal conductivity depends on the temperature, an analytical solution of the linear equation is obtained, to make a graphic comparison with the nonlinear one. This will be applied to a bar of length* $0 \leq x \leq L = 10\ cm$*, which is initially at a uniform temperature* $T(0,t) = 0\ ^oC$ *and for x=L;* $T(L,0) = 100\ ^oC$*, where it is assumed that the thermal conductivity depends on the temperature linearly:* $K(T) = K_0(1 + \beta T)$*.*

***Results**: It is observed that the difference between the linear equation with the non-line where there are changes between them in temperature, another approximation could be made when the degree of non-linearity is increased.*

***Conclusion:** It is perceived that there is a significant increase in temperature in the nodes. Considering the temperature values, an indication is noted that the system is going to stabilize. For the case in which the conductivity is highly nonlinear, but K(T) admits a Taylor series expansion, the Kirchhoff transform is more complicated and the substitution in the nonlinear equation leads again to a nonlinear equation.*

***Key Word**: Heat Equation, Kirchhoff, Poisson Equation, Nonlinear Heat Equation, Temperature, Thermal Conductivity*

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I. Introduction

There are an infinity of nonlinear partial and ordinary differential equations in which it is difficult to find a transformation or perform a Taylor expansion around a point in question to take it to a linear form and thus be able to solve it.

II. Methodology

The Gustav Robert Kirchhoff transformation is used to linearize the nonlinear equation, this nonlinearity is presented when the thermal conductivity depends on the temperature, you get analytic solution of the linear equation, to perform a graphical comparison with the nonlinear.

We want to find numerically the integral in an interval given by Equation (1)

The equation that models the phenomenon of heat transfer with variable thermal conductivity is found in [1-5] and is given by:

$$\rho C_p \frac{\partial T}{\partial t} = \nabla(K(T)\nabla T) + g \tag{1}$$

where ρ, C_p, K density, specific heat and thermal conductivity, which are functions of temperature, and the term source of heat generation is independent of temperature $\mathrm{g} = \mathrm{g}(\vec{r}, t)$.

In order to solve this problem, we proceed as follows. The transformation *K(T)* is defined using the property given in [5]:

$$U = U(T) = \int_0^T \frac{K(T')}{K_0} dT' \tag{2}$$

$\mathrm{T} = \mathrm{T}(\vec{r}, t)$, where
K_o is the value of thermal conductivity for *t=0*.

Equation (2) is called the Gustav Robert Kirchhoff transformation. Where *K(T)* is a function of temperature then Equation (1), can be written as:

$$\rho C_p \frac{\partial T}{\partial t} = K(T)\nabla^2 T + \nabla K(T)\nabla T + g \tag{3}$$

where it is obtained from

$$\nabla K(T) = \frac{dK(T)}{dT}\nabla T \tag{4}$$

Substituting Equation (4) in Equation (1), is obtained:

$$\rho C_p \frac{\partial T}{\partial t} = K(T)\nabla^2 T + \frac{dK(T)}{dT}(\nabla T)^2 + g \tag{5}$$

In order to simplify Equation (5) using Equation (2) and Equation (4), for the function *U(T)* from *K(T)* we proceed as follows, from the Leibnitz rule we have

$$\frac{d}{dt}\left(\int_{f(t)}^{g(t)} F(\xi, t) d\xi\right) = \int_{f(t)}^{g(t)} \frac{\partial F(\xi, t)}{\partial t} d\xi + g'(t)F(g(t), t)) - f'(t)F(f(t), t) \tag{6}$$

applying it Equation (6) to Equation (2) we have

$$\frac{d}{dt}U(T) = \frac{d}{dt}\left(\int_0^T \frac{K(T')}{K_o} dT'\right) = \frac{1}{K_o}\int_0^T \frac{\partial K(T')}{\partial t} dT' + \frac{K(T)}{K_o}\frac{dT}{dt} - (0)\frac{K(0)}{K_0} \tag{7}$$

where $\frac{dK(T)}{dt} = 0$ then

$$\frac{\partial U(T)}{\partial t} = \frac{K(T)}{K_o}\frac{dT}{dt} \tag{8}$$

and from the fundamental theorem of integral calculus we obtain:

$$\frac{d}{dt}\left(\int_0^T \frac{K(T')}{K_o} dT'\right) = \frac{K(T)}{K_o} \tag{9}$$

resulting

$$\nabla U(T) = \frac{dU(T)}{dT}\nabla T = \frac{K(T)}{K_o}\nabla T \tag{10}$$

$$\nabla^2 U(T) = \nabla\left[\frac{K(T)}{K_o}\nabla T\right] = \frac{1}{K_o}[\nabla K(T)\nabla T + K(T)\nabla^2 T] \tag{11}$$

after Equation (4) we have

$$\nabla^2 U(T) = \frac{1}{K_o}\left[\frac{dK(T)}{dt}\nabla T\, \nabla T + K(T)\nabla^2 T\right] \tag{12}$$

$$\nabla^2 U(T) = \frac{1}{K_o}\left[\frac{dK(T)}{dt}(\nabla T)^2 \ + K(T)\nabla^2 T\right] \tag{13}$$

and substituting Equation (8) and Equation (7) in Equation (5), we obtain

$$\rho C_p \frac{K_o}{K(T)}\frac{\partial U(T)}{\partial t} = K_o \nabla^2 U(T) + g \tag{14}$$

$$\frac{\rho C_p}{K(T)}\frac{\partial U(T)}{\partial t} = \nabla^2 U(T) + \frac{g}{K_o} \tag{15}$$

$$\frac{1}{\alpha}\frac{\partial U(T)}{\partial t} = \nabla^2 U(T) + \frac{g}{K_o} \tag{16}$$

$\alpha = \frac{K(T)}{\rho C_p}$ finally remaining

$$\frac{1}{\alpha}\frac{\partial U(T)}{\partial t} = \alpha \nabla^2 U(T) + \frac{\alpha}{K_o} g \tag{17}$$

where the thermal diffusivity α is a function of temperature [5]. Equation (11) is a simpler equation in its structure, since it is assumed that the variation of thermal diffusivity, with respect to temperature, is negligible, therefore it is an almost linear equation that can be solved without major problems.

For example, taking the values of $\alpha = cte$; $\alpha = 10$ cm^2/seg, with $g = 0$ we have the problem in normal form, and using the given Gustav Robert Kirchhoff transform we have:

$$\frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t} = \frac{\partial^2 T(x,t)}{\partial x^2} \quad 0 \leq x \leq 10 \tag{18}$$

under frontier conditions

$$T(0,t) = T_o = 0^o C \quad t>0 \tag{19}$$

$$T(10,t) = T_1 = 100^o C \quad t>0 \tag{20}$$

and initial condition

$$T(x,0) = 0^o C \text{ for } t = 0 \tag{21}$$

On the other hand, solving Equation (12) by the method of separation of separable variables we have [6-10]:

$$T(x,t) = 100\begin{bmatrix} \frac{x}{10} + \\ 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi} sen\left(\frac{n\pi x}{10}\right) e^{-\frac{n^2\pi^2 t}{10}} \end{bmatrix} \tag{22}$$

For the second case when $g=0$ and α the thermal diffusivity varies linearly with respect to the temperature Equation (11), we have:

$$\frac{1}{\alpha}\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \quad 0 \leq x \leq 10 \tag{23}$$

under frontier conditions

$$U(0,t) = U_o = 0^0 C \tag{24}$$

$$U(L,t) = U_1 = 100 + \frac{\beta}{2}(100)^2 = 100(1+50\beta) \tag{25}$$

and initial condition

$$U(x,0) = 0^o C \tag{26}$$

whose solution is:

$$U(x,t) = 100\left[\frac{x}{10}(1+50\beta) + 2(1+50\beta)\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}sen\left(\frac{n\pi x}{10}\right)e^{-\frac{n^2\pi^2 t}{10}}\right] \quad (27)$$

then the transformation from $U(x,t)$ to $T(x,t)$ will be given as follows [5]:

$$T(x,t) = \frac{1}{\beta}\left[\sqrt{1+2\beta U(x,t)} - 1\right]$$

$$T(x,t) = \frac{1}{\beta}\left[\sqrt{1+200\beta\left[\frac{x}{10}(1+50\beta) + 2(1+50\beta)\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}sen\left(\frac{n\pi x}{10}\right)e^{\frac{-n^2\pi^2 t}{10}}\right]} - 1\right] \quad (28)$$

For a particular case, the values of α=10, β=0.1 are taken, these are substituted in Equation (21), we have the expression of the temperature as

$$T(x,t) = 100\left[\frac{x}{10} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}sen\left(\frac{n\pi x}{10}\right)e^{-\frac{n^2\pi^2 t}{10}}\right] \quad (29)$$

III. Evaluations

The following tables are shown evaluating Equation (21), for times *t=2,2.4,6*.

Table no 1: Temperature distribution for t=2 seconds.

Distance in cm	Temperature in degrees
0	0
1	7.2742
2	14.8133
3	22.569
4	31.5995
5	41.1567
6	51.5826
7	62.8344
8	74.7908
9	87.2604
10	100

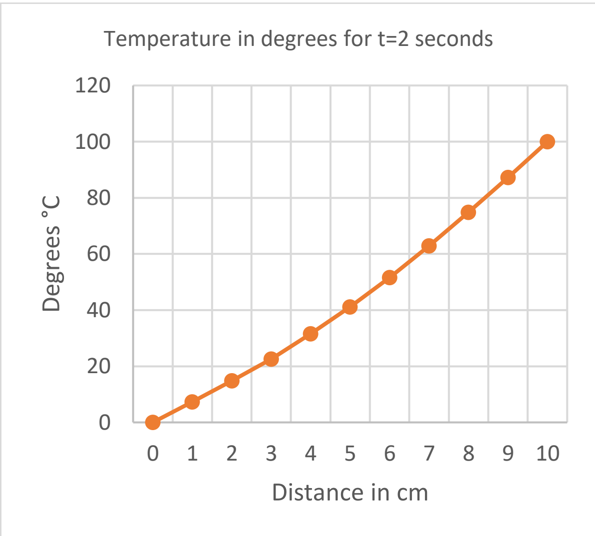

Fig. 1. Shows the solution of the linear equation for *t=2* seconds

Table no 2: Temperature distribution for t=2.4.

Distance in cm	Temperature in degrees
0	0
1	8.1601
2	16.4998
3	25.1815
4	34.3343
5	44.0412
6	54.3314
7	65.1769
8	76.4952
9	88.1572
10	100

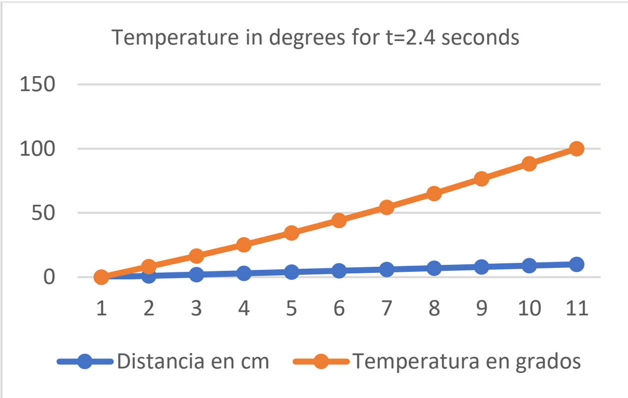

Fig. 2. Shows the solution of the linear equation for *t=2.4* seconds

Table no 3: Temperature distribution for t=6 seconds.

Distance in cm	Temperature in degrees
0	0
1	7.2742
2	14.8133
3	228569
4	31.5995
5	41.1567
6	51.5826
7	62.8344
8	74.7908
9	87.2604
10	100

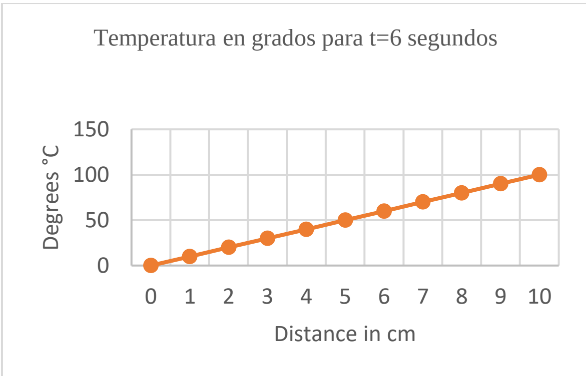

Fig. 3. Shows the solution of the linear equation for *t=6* seconds

For the case when the thermal conductivity is variable, we have the series (13) up to the first three terms:

$$T(x,t) = \frac{1}{\beta}\left[\sqrt{\begin{matrix}1+200\beta \\ \left[\frac{x}{10}(1+50\beta)+2(1+50\beta)\right] \\ \sum_{n=1}^{\infty}\frac{(-1)^n}{n\pi}\, sen\left(\frac{n\pi x}{10}\right) e^{\frac{-n^2\pi^2 t}{10}}\end{matrix}} - 1\right] \quad (28)$$

The tables are shown below using Equation (15), for times *t=2, 2.4,6*.

Table no 4: Temperature distribution for t=2 seconds.

Distance in cm	Temperature in degrees
0	0
1	21.1915
2	33.3313
3	43.3182
4	52.3826
5	60.9846
6	69.309
7	77.4078
8	85.2633
9	92.8166
10	100

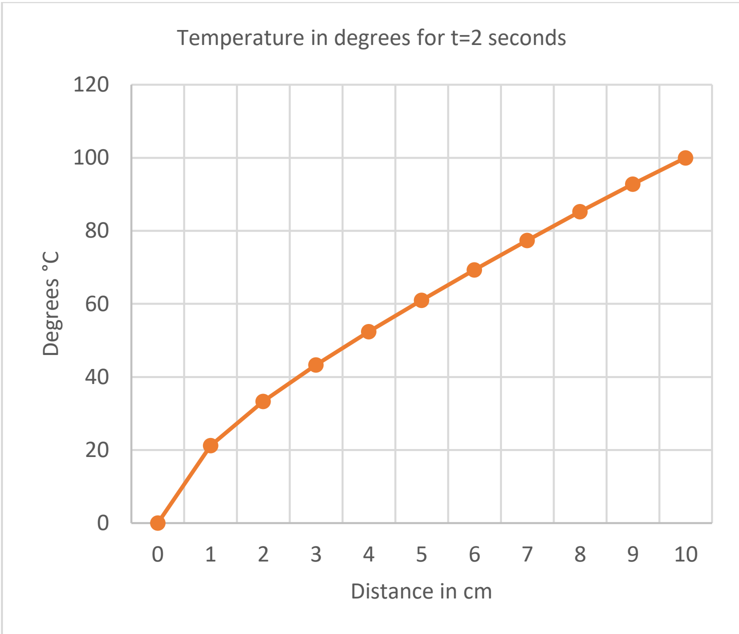

Fig. 4. Shows the solution of the linear equation for *t=2* seconds

Table no 5: Temperature distribution for t=2.4.

Distance in cm	Temperature in degrees
0	0
1	22.8513
2	35.6068
3	45.8729
4	54.9624
5	63.3822
6	71.362
7	79.0013
8	86.3298
9	93.3386
10	100

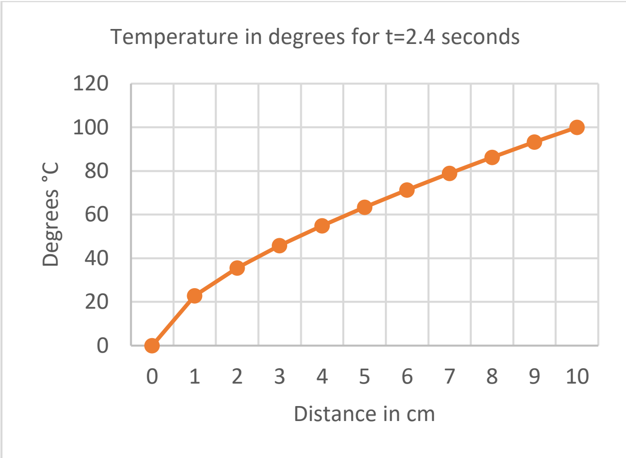

Fig. 5. Shows the solution of the linear equation for *t=2.4* seconds

Table no 6: Temperature distribution for t=6.

Distance in cm	Temperature in degrees
0	0
1	25.9677
2	39.8795
3	59.6913
4	59.8608
5	67.9713
6	75.326
7	82.1056
8	88.4275
9	94.3728
10	100

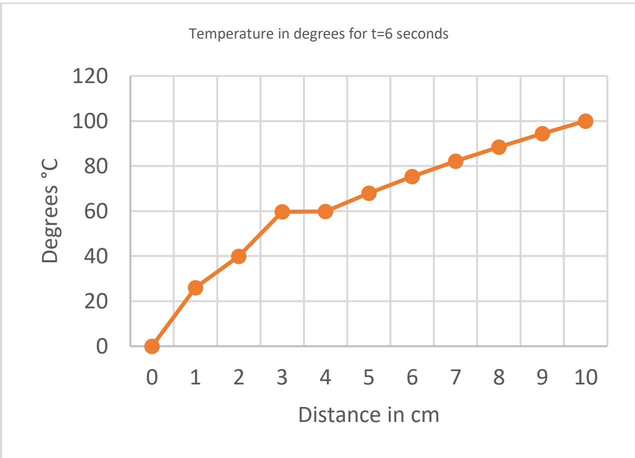

Fig. 6. Shows the solution of the linear equation for *t=6* seconds

Comparison between linear and nonlinear equation

A comparison of the linear and non-linear equation will be made.

Table no 7: Shows the results of Equations 14-15 for t=2 seconds.

Distance in cm	Equation 27	Equation 28
0	0	0
1	7.2742	21.1915
2	14.8133	33.3313
3	22.569	43.3182
4	31.5995	52.3826
5	41.1567	60.9846
6	51.5826	69.309
7	62.8344	77.4078
8	74.7908	85.2633
9	87.2604	92.8166

10	100	100

The graphs of the linear and non-linear equation are shown to observe their behavior.

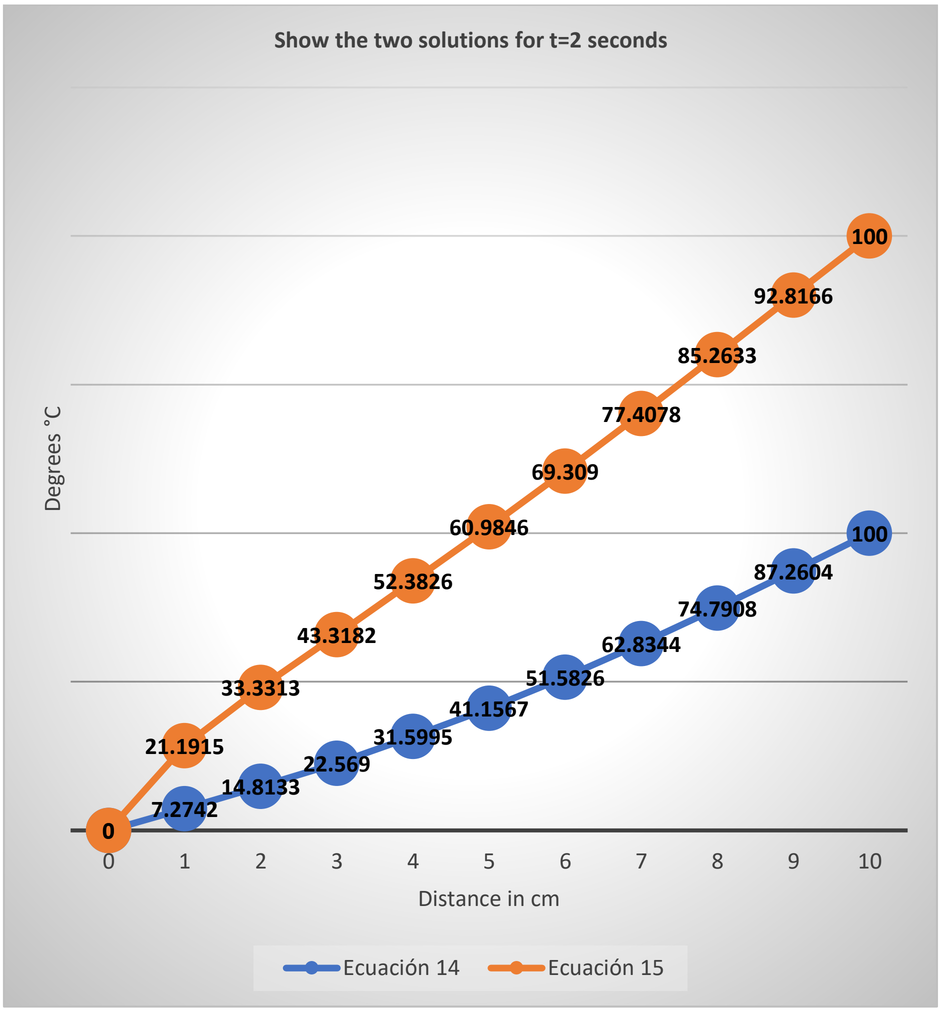

Fig. 7. Temperature distribution from Equation 14-15 for *t=2* seconds

Table no 8: Shows the results of Equations 14-15 for t=2.4 seconds.

Distance in cm	Equation 27	Equation 28
0	0	0
1	8.1601	22.8513
2	16.4998	35.6068
3	25.1815	45.8729
4	34.3343	54.9624
5	44.0412	63.3822
6	54.3314	71.362
7	65.1769	79.0013
8	76.4952	86.3298
9	88.1572	93.3386
10	100	100

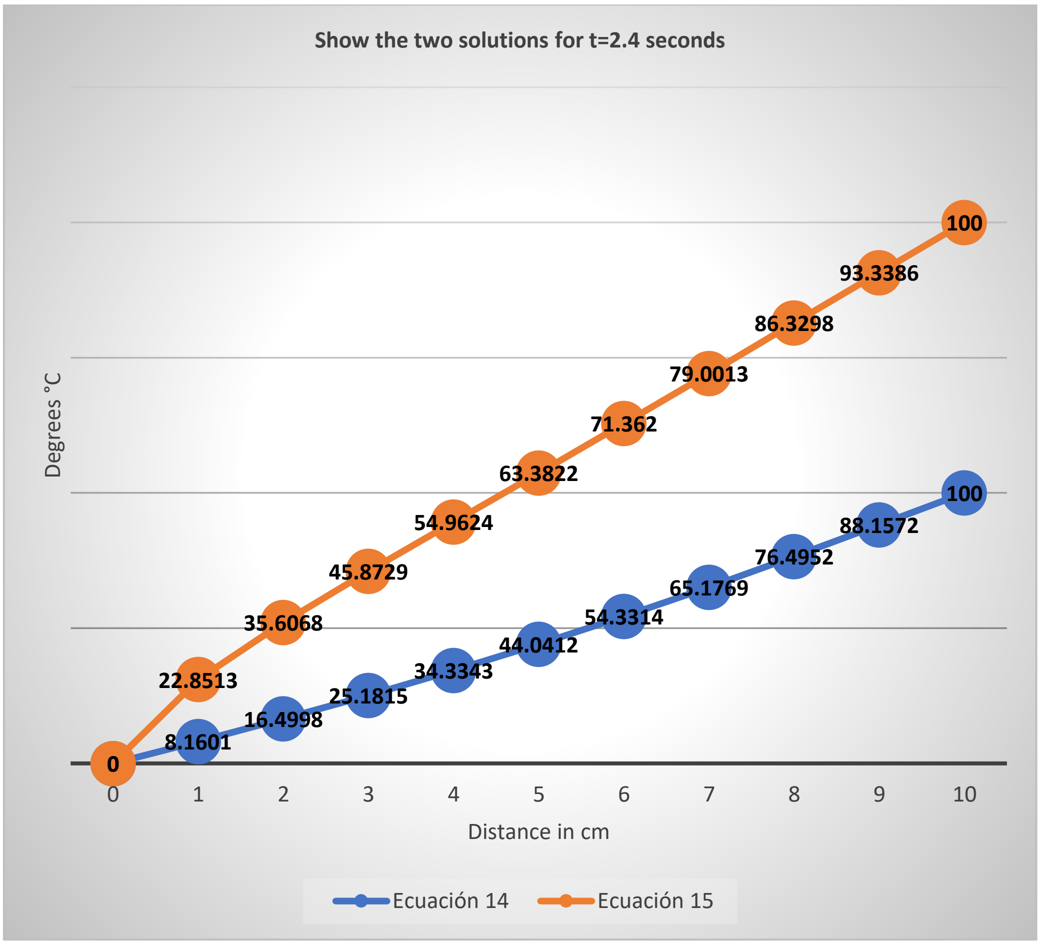

Fig. 7. Temperature distribution from Equation 14-15 for *t=2.4* seconds

Table no 9: Shows the results of Equations 14-15 for t=6 seconds.

Distance in cm	Equation 27	Equation 28
0	0	0
1	9.9473	25.9677
2	19.8997	39.8795
3	29.8619	59.6913
4	39.8377	59.8608
5	49.8294	67.9713
6	59.8377	75.326
7	69.862	82.1056
8	79.8997	88.4275
9	89.9473	94.3728
10	100	100

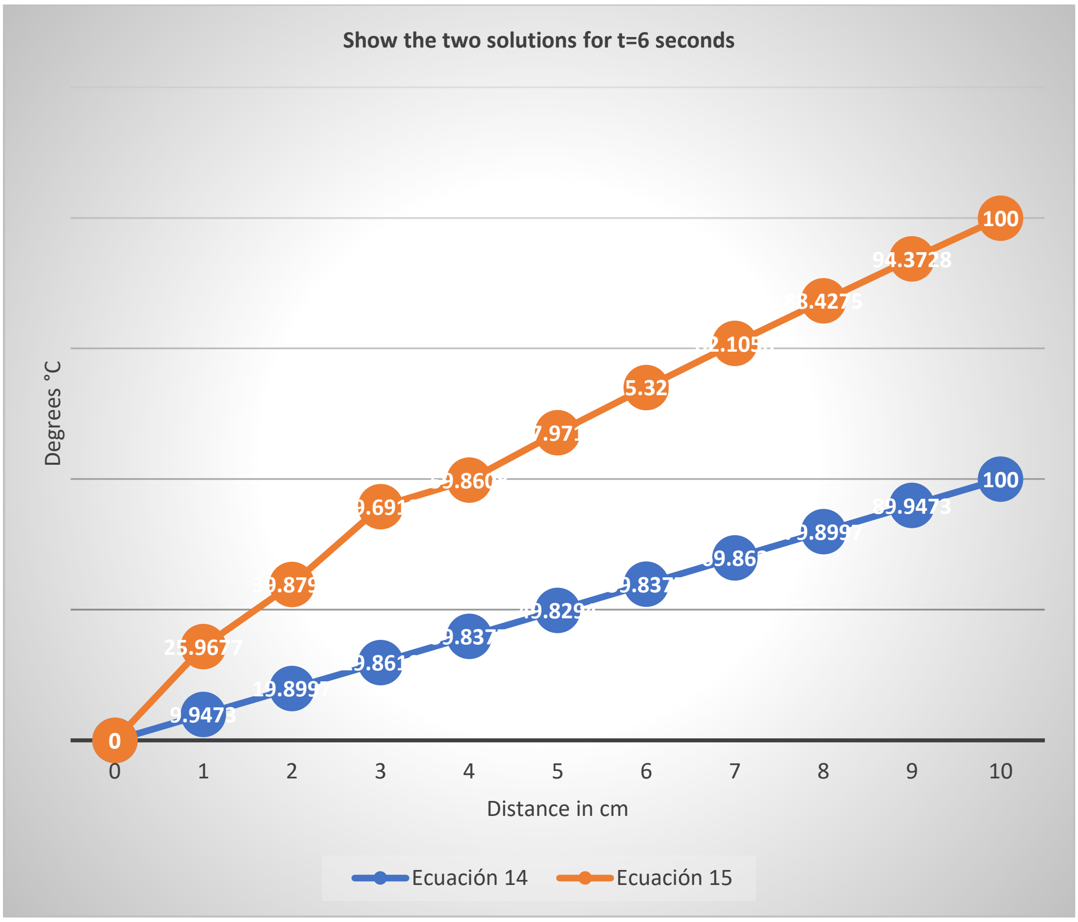

Fig. 9. Temperature distribution from Equations 14-15 for *t=6* seconds

IV. Results and Discussion

In Fig. 7, 8 and 9 it is clearly shown how the difference that exists in the linear equation with the non-line where there are changes in temperature between them, another approximation could be made when the degree of non-linearity is increased.

V. Conclusion

From the graphs it can be seen that there is a significant increase in temperature in the nodes. Observing the temperature values shows an indication that the system is going to stabilize. For the case in which the conductivity is highly nonlinear, but *K(T)* admits a Taylor series expansion, the Kirchhoff transform is more complicated and the substitution in the nonlinear equation leads again to a nonlinear equation.

$$K(T) = K_0 \sum_{n=0}^{n} \beta_n (T - T_0)^n$$

It remains as a case study when *K(T)* is of higher order than the first.

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