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A New Approach ToNewton's Method Using The Intermediate Rule For The Riemann-StieltjesSums

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Abstract:
A modification to the iterative Newton's method to find roots of a non-linear equation is reported. Newton's method is used in physical-mathematical sciences, engineering, biology, medicine, and many other disciplines. The recent modification consists in using an indefinite integral in the algorithm, which is approximated by the intermediate point rule of the Riemann-Stieltjes' sums. It has been demonstrated that this method exhibits cubic convergence order. A computer program was also developed to find the roots of a few sample equations which were compared to the real approximate solutions. The order of the convergence of the iterative Newton's method is quadratic, while other methods that use numerical integration, such as the trapezoidal approach, have a third-order convergence. However, methods employing the intermediate point rule of Riemann-Stieltjes can achieve at least cubic convergence.
Background*:Since the 19th century, one of the most popular problems in mathematics has been the numerical methods used to find approximate solutions to non-linear equations. These methods are applied not only in this field but also in basic sciences, engineering, biological sciences, and medicine.*
Materials and Methods*: The purpose of this study is to derive an alternative numerical method for calculating the root of a non-linear equation, using Newton's method as a basis. This involves approximating the derivative of the function using the rectangle method and subsequently determining its order of convergence.*
Results*: The mathematical proof of convergence shows that the method pertains to the first order of convergence, but it exhibits behavior similar to that of a third order of convergence.*
Conclusion:*The method obtained was tested in three applications, and accurate results were achieved. Therefore, it can be utilized to find a single root of nonlinear equations.*
Key Word*:Newton's method, Trapezoidal method, Convergence, Convergence Index, Midpoint rule of rectangles*

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I. Introduction

When using numerical analysis, it is possible to employ several iterative methods to find a root of a non-linear equation.The Newton's method is the most popular method used to solve Equation (1).
$f(x) = 0(1)$

The efficiency index, EI, in the iteration is defined as $EI = \rho^{\frac{1}{v}}$.Gautschi[1], as well as Grau-Sánchez and others[2], propose a method to calculate this index. Newton's method is of quadratic order with an efficiency index of$EI \approx 1.414213$.

Weekeron and Fernando[3]use the trapezoidal method in their formulation to solveEquation (1), reaching an efficiency of$EI \approx 1.442249$. In this paper, the intermediate point of the rectangles is employedto evaluate the Riemann-Stieltjes integrals to reach a cubic order of convergence.

II. Description of the proposed algorithm.

The first base modified method used the trapezoidal approach to evaluate the integrals. In this paper, the intermediate point to evaluate the Riemann-Stieltjes sums is incorporated.
By applying the Newton's theorem, the Weerakoon and Fernando's formulation is obtained[3].
$f(x) = f(x_n) + \int_{x_n}^{x} f'(\lambda)d\lambda(2)$
Equation (2) uses the trapezoidal integration method, leading to the Weerakoon and Fernando's algorithm. The latter is explained afterwards.
Weerakoon and Fernando's algorithm.

Assuming that x_0 is the initial value to start the iterations, x_{n+1} can be obtained as shown in equation (3).

$$x_{n+1} = x_n - \left(\frac{f(x_n)}{f(x_n)+f'(x^*_{n+1})}\right)(3)$$

In equation (3), $x^*_{n+1)}$ is defined as shown in equation (4).

$$x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}(4)$$

The convergence criterion to stop the iteration can be considered as:if $|x_{n+1} - x_n| \leq \varepsilon$, $|f(x_n)| \leq \varepsilon$, where ε is set at the beginning of the iteration.

Using the method proposed in this paper, the integral in Equation (2) is approximated using the intermediate point of Riemann- Stieltjes sums.

Figure 1 graphically illustrates the elements used to numerically evaluate the integral once the intermediate point of the Riemann-Stieltjes sumsis applied.

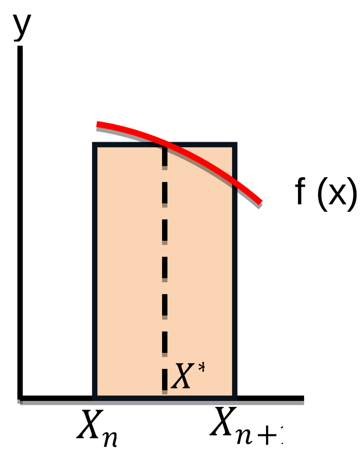

Fig. 1. The integration of the rectangle using the intermediate point.

$$\int_{x_n}^{x_{n+1}} f'(x)\,dx \approx (x_{n+1} - x_n) f'\left(\frac{x_{n+1}+x_n}{2}\right)(5)$$

The error using the intermediate point of the Ashis Kumar Dash's rectangles[4] isobtained by using Equation (6).

$$E_{middle\ \ point} = -f'(\xi)\frac{(x_{n+1}-x_n)^3}{12}, \xi \in (x_n, x_{n+1})(6)$$

where

$$x^* = \frac{x_n + x_{n+1}}{2}(7)$$

For only one intermediate point, TchavdarMarinov[5]found the evaluation of the error of the numerical integration. Marinov concluded that if the function $f(x)$ has a continuous second derivative, the error using the intermediate point rule can be approximated by equation (8).

$$E_{middle\ \ point} = -f''(\xi)\frac{(x_{n+1}-x_n)^3}{24},\ \xi \in (x_n, x_{n+1})\ (8)$$

By applying the result of Equation (3) to the integral in Equation (2), Equation (9)is obtained.

$$\int_{x_n}^{x} f'(\lambda)d\lambda \approx (x - x_n) f'\left(\frac{x+x_n}{2}\right)(9)$$

In each iteration of Newton's method, the tangent straight line at the intermediate point is given by Equation (10).

$$M_n(x) = f(x_n) + f'\left(\frac{x+x_n}{2}\right)(x - x_n)(10)$$

Therefore, the next iterative point x_{n+1} is calculated as the root of the local model given by Equation (11).

$$M_n(x_{n+1}) = 0(11)$$

Consequently, the equation to find the root can be written as shown in Equation (12).

$$0 = f(x_n) + (x_{n+1} - x_n) f'\left(\frac{x_{n+1}+x_n}{2}\right)(12)$$

Once x_{n+1} is solved in Equation (12),Equation (13) is obtained.

$$x_{n+1} = x_n - \left(\frac{f(x_n)}{f'\left(\frac{x^*_{n+1}+x_n}{2}\right)}\right)(13)$$

$$x^*_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}(14)$$

Equation (13) formulates another iterative technique of Newton's method using the intermediate point of the Riemann- Stieltjes sums.
Assuming that

$x_{n+1}^{*} = x_n - \frac{f(x_n)}{f'(x_n)} = z_n$(15)

And

$y_n = \frac{x_n + x_{n+1}^{*}}{2}$(16)

then, by substituting these two variables in Equation (13), a three-step iterative algorithm is obtained, which is described subsequently.

Algorithm.
Given an initial value of x_0, the value of y is calculated, and the derivative is substituted to obtain an approximate value of x_{n+1}. The iteration algorithm is given by Equations (17), (18) and (19).

$x_{n+1} = x_n - \left(\frac{f(z_n)}{f'(y_n)}\right)$(17)

$z_n = x_n - \frac{f(x_n)}{f'(x_n)}$(18)

$y_n = \frac{z_n + x_n}{2}$(19)

For $n = 0, 1, 2$ ...
The stopping criterion for the iterations is $|x_{n+1} - x_n| \leq \varepsilon$.
The algorithm ends when the approximated root α, is obtained:$\alpha \approx x_{n+1}$, because of $f(x_{n+1}) \approx 0$.

III. Convergence analysis.

Let$f\epsilon C^{n+1}(I)$be a nonlinear equation, and $\alpha\epsilon I$asimple rootin an open interval of I. For the root , $f(\alpha) = 0$. The Taylor expansion of f around $x_n$is shown in Equation (20).

$f(x) = f(x_n) + \frac{f'(x_n)}{1!}(x - x_n) + \frac{f''(x_n)}{2!}(x - x_n)^2 + \cdots + \frac{f^{(n)}(x_n)}{n!}(x - x_n)^n$(20)

If $x_0 \in (\alpha - \delta, \alpha + \delta)$ with $\delta > 0$, and if a simple root exists in this interval, then the convergence of this method is guaranteed.

Theorem. Let be $\alpha\epsilon I$, $f\epsilon C^{n+1}(I)$ where $f: I \subset \mathbb{R} \to \mathbb{R}$ exists in an interval I, then each of the Equations (17), (18), and(19), is of first order convergence.
Proof. Let α be a simple root of f, so that $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Defining$e_n = x_n - \alpha$, as f is of class $C^{n+1}(I)$ then $f(x_n)$ can be expanded in a Taylor series around α.

$f(x_n) = f(\alpha + e_n) = f(\alpha) + \frac{f'(\alpha)}{1!}(x_n - \alpha) + \frac{f''(\alpha)}{2!}(x_n - \alpha)^2 + \cdots + \frac{f^{(n)}(\alpha)}{n!}(x_n - \alpha)^n$(21)

As $f(\alpha) = 0$, by substituting Equation(21), Equation (22)is obtained.

$f(x_n) = 0 + \frac{f'(\alpha)}{1!}e_n + \frac{f''(\alpha)}{2!}e_n^2 + \frac{f'''(\alpha)}{3!}e_n^3 + O(e_n^4)$(22)

By factorizing $f'(\alpha)$, from Equation(22), Equation (23) is obtained.

$f(x_n) = f'(\alpha)\left(e_n + \frac{f''(\alpha)}{2!f'(\alpha)}e_n^2 + \frac{f'''(\alpha)}{3!f'(\alpha)}e_n^3\right) + O(e_n^4)$(23)

The following constantsare defined: $c_2 = \frac{f''(\alpha)}{2!f'(\alpha)}$, $c_3 = \frac{f'''(\alpha)}{3!f'(\alpha)}$, which are then substituted in Equation (23). The generalization of the last constants can be written as:

$c_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)}$ para $k = 2,3$ ...(24)

Thus, Equation (23)can be written as:

$f(x_n) = f'(\alpha)(e_n + c_2 e_n^2 + c_3 e_n^3) + O(e_n^4)$(25)

By taking the derivative of Equation(25)with respect to e_n, the Equation(26) is obtained.

$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n e_n^2 + 3c_3 e_n^2 + O(e_n^3)]$(26)

By dividing Equation (26) by Equation(26), Equation(27) is obtained.

$\frac{f(x_n)}{f'(x_n)} = [e_n + c_2 e_n^2 + c_3 e_n^3]\{1 - [2c_2 e_n + 3c_3 e_n^2 + O(e_n^3)]\}^{-1}$(27)

By using the geometric series $\frac{1}{1-x} = 1 - x + x^2 - x^3, \ldots,$ which converges only if $|x| < 1$, Equation (27) is obtained from Equation (28).

$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + (2c_2^2 - 2c_3)e_n^3 + O(e_n^4)$(28)

In Equation (23),$e_n = x_n - \alpha$, and by solving for $x_n = \alpha + e_n$, Equation (29) is obtained.

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + e_n - \left(e_n - c_2 e_n^2 + (2c_2^2 - 2c_3)e_n^3 + O(e_n^4)\right)$ (29)

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + O(e_n^4)$ (30)

Defining the variable z_n in Equation (30).

$z_n = \alpha + c_2 e_n^2 + (2c_3 + 2c_2^2)e_n^3 + O(e_n^4)$ (31)

By substituting the values of x_n and of z_n into Equation(17), Equation (32) is obtained.

$y_n = \alpha + \frac{1}{2}e_n + \frac{1}{2}c_2 e_n^2 + \frac{1}{2}(2c_3 - 2c_2^2)e_n^3 + O(e_n^4)$ (32)

Afterwards, Equations (23) and (25) are substituted into Equation (9).

By carrying out the algebra and simplifying, the Equation(33)can be written as:

$z_n = \alpha + c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + O(e_n^4)$(33)

By expanding $f(z_n)$and $f(y_n)$in Taylor series around α and using that $(\alpha) = 0$, Equation (34) is obtained.

$f(z_n) = f'(\alpha)\left(c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + O(e_n^4)\right)$(34)

This leads to Equation (35).

$f(y_n) = f'(\alpha)\left(\frac{1}{2}e_n + \frac{3}{4}c_2 e_n^2 + \frac{1}{2}(c_3 - c_2^2)e_n^3 + O(e_n^4)\right)$(35)

By taking the derivates of $f(z_n)$ and $f(y_n)$ with respect to e_n, Equations(30), (31), (32), (34), and(35) are obtained.

$f'(z_n) = f'(\alpha)\left(2c_2 e_n + 6(c_3 - c_2^2)e_n^2 + O(e_n^3)\right)$(36)

$f'(y_n) = f'(\alpha)\left(\frac{1}{2} + \frac{3}{2}c_2 e_n + \frac{3}{2}(c_3 - c_2^2)e_n^2 + O(e_n^3)\right)$(37)

$f'(y_n) = \frac{1}{2}f'(\alpha)\left(1 + 3c_2 e_n + 3(c_3 - c_2^2)e_n^2 + O(e_n^3)\right)$(38)

$\frac{f(x_n)}{f'(y_n)} = \frac{f'(\alpha)(e_n + c_2 e_n^2 + c_3 e_n^3) + O(e_n^4)}{\frac{1}{2}f'(\alpha)\left(1 + 3c_2 e_n + 3(c_3 - c_2^2)e_n^2 + O(e_n^3)\right)} = 2[(e_n + c_2 e_n^2 + c_3 e_n^3) + O(e_n^4)]\left[\left(1 + 3c_2 e_n + 3(c_3 - c_2^2)e_n^2 + \right.\right.$

$Oen3-1$(39)

$\frac{f(x_n)}{f'(y_n)} = \frac{f'(\alpha)(e_n + c_2 e_n^2 + c_3 e_n^3) + O(e_n^4)}{\frac{1}{2}f'(\alpha)\left(1 + 3c_2 e_n + 3(c_3 - c_2^2)e_n^2 + O(e_n^3)\right)} = 2[(e_n + c_2 e_n^2 + c_3 e_n^3) + O(e_n^4)]\left[\left(1 + 3c_2 e_n + 3(c_3 - c_2^2)e_n^2 + \right.\right.$

$Oen3-1$ (40)

By applying the geometrical series given in Equation(39), Equations (41),(42) and(43) are obtained.

$\frac{f(x_n)}{f'(y_n)} == 2[(e_n - c_2 e_n^2 + 3(3c_3 + c_2^2)e_n^3) + O(e_n^4)]$(41)

$x_{n+1} = x_n - \frac{f(x_n)}{f'(y_n)}$(42)

$e_{n+1} + \alpha = e_n + \alpha - 2[(e_n - c_2 e_n^2 + 3(3c_3 + c_2^2)e_n^3) + O(e_n^4)]$ (43)

$e_{n+1} = -e_n - c_2 e_n^2 + 6(3c_3 + c_2^2)e_n^3 + O(e_n^4)$(44)

Equation (44) establishes the first order convergence.Afterwards, the three steps are introduced for the proposed method, where Newton's method has quadratic convergence. Therefore, the convergence rate is slower for the Weerkoon and Fernando's method, which has cubic convergence. Thus, the method using the intermediate point for the Riemann- Stieltjes sums also exhibits cubic convergence.

The efficiency index for this method is given by:

$\rho = 3$, $v = 3$which results in$EI \approx 1.444224957$.

IV. Sample applications

Three applications are shown of the intermediate point using the rectangle method, and a comparison is made with the results of the Weerkoon and Fernando's trapezoidal method. A stopping criterion with $\varepsilon = 10^{-5}$ is used.

Let us find the root of$f(x) = x^2 - 2$ with an initial value of $x_0 = 1$. $\alpha = 1.41421356$ is considered as the exact real solution. The results of the iterations are shown in Table no 1.

Table no 1. Solution and error in solving $(x) = x^2 - 2$.

Iteration	Intermediate point of rectangle method	Absolute error of the intermediate point of the rectangle	Weerkoon and Fernando's Trapezoidal method	Absolute error of the Weerkoon and Fernando's trapezoidal method
1	1.4	0.0142136	1.4	0.0142136
2	1.4142132	0.0000004	1.4142132	0.0000004
3	1.4142135	0	1.4142135	0

Let us consider the function $f(x) = (x-1)^3 - 1$ with an initial value of $x_0 = 2.5. \alpha = 2$is considered as the true real solution. The results of the iterations are shown in Table no 2.

Table no 2. Solution and error in solving $f(x) = (x-1)^3 - 1$.

Iteration	Intermediate point rectangle method	Absolute error of the intermediate point rectangle method	Weerkoon and Fernando's trapezoidal method	Absolute error of the Weerkoon and Fernando's trapezoidal method
1	2.0484376	0.0484376	2.0562711	0.0562711
2	2.0000951	0.0000951	2.0001843	0.0001843
3	2	0	2	0

Let us consider the function $f(x) = (x-1)^3 - 1$, with an initial value of $x_0 = 3.5$. $\alpha = 2$ was considered as the true real solution. The results of the iteration are shown in Table no 3.

Table no 3. Solution and error in solving equation $f(x) = (x-1)^3 - 1$.

Iteration		Intermediate point rectangle method	Absolute error of the intermediate point rectangle method	Weerkoon and Fernando's trapezoidal method	Absolute error of the Weerkoon and Fernando's trapezoidal method
1		2.4050112	0.4050112	2.4411840	0.4411840
2		2.0298204	0.0298204	2.0426252	0.0426252
3		2.0000229	0.0000229	2.0000825	0.0000825
4		2	0	0	0

V. Discussion

The efficiency indexes of the two methods have been analyzed.Using newton's method, with $\rho = 2$ and $v = 2, EI \approx 1.414213$ is obtained.For Newton's method using the intermediate point rule for the Riemann-Stieltjes sums, with

$\rho = 3$ and $v = 3, EI \approx 1$ is obtained. The difference between the two indexes is of: 0.03111957.Therefore, it isconcluded that the optimalstrategyentails employing Newton's method with the intermediate point approach for evaluating the Riemann-Stieltjes sums.

VI. Conclusion

Firstly, it is necessaryto use the Newton's method to consider the trapezoidal and the intermediate point rectangle rule methods. These two methods are applied to solve sample equations using a numerical scheme through C++ language to find the absolute errors as well as the approximate or exact solution. For the intermediate point rectangle method, a smaller error was obtained than the one using the Weerkoon and Fernando's trapezoidal method.

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