IOSR Journal of Applied Physics (IOSR-JAP)
e-ISSN: 2278-4861.Volume 6, Issue 6 Ver. III (Nov.-Dec. 2014), PP 17-28
www.iosrjournals.org

A Simple yet Elegant Expansion Method to Solve Radiative Transport Problems in Finite Media

H.F. Machali[1], M.H. Haggag[1, 2]
[1]*(Physics Department, Faculty of Science, Mansoura University, Mansoura, Egypt)*
[2]*(Physics Department, Faculty of Science, Taif University, Taif 888, K.S. A.)*

Abstract: *Radiative heat transfer in one dimension is studied in a plane – parallel geometry for an absorbing and isotropically scattering medium subjected to azimuthally symmetric incident radiation at boundaries. The integral form of the transport equation, which is weakly singular Fredholm integral equation of the second kind, is used .The unknown function in the integral form is expanded in terms of truncated Chebyshev polynomials in the optical variable. The collocation method is applied to obtain a system of linear algebraic equations for the expansion coefficients. Numerical calculations are done for the transmissivctivity, and exit angular distributions of slabs with various values of single scattering albedo. Comparisons between the present and available results in references indicate that our results are accurate, as shown in the tables.*

Keywords**:** *Radiative transfer, Isotropic scattering, planar slab, Reflectivity and transmissivity, Chebyshev polynomials,* Collocation method.

I. Introduction

The transfer of heat which is due to thermal radiation is referred to as radiative transfer and is an important heat transfer process for high temperature applications such as energy conversion systems , nuclear reactors , many industrial processes and solar energy conversion devices . Radiation has complicated transfer mechanisms that are difficult to model even in a simple system. Early investigations predicted one dimensional radiative transfer and neutron transport in planar media using various solution techniques [1- 19]. Among these various techniques, the so called expansion technique in which suitable expansion of the radiation intensity in the angular variable is used to solve the integro – differential form of the transport equation [20 –29]. Other methods [30- 33] have been developed in the context of an integral formulation of the transport equation where Fredholm integral equations of the second kind are produced in terms of the Legendre moments of the total intensity. Power series expansions in the optical variable have been used often in this context [30, 31]. Legendre and Chebyshev polynomials both of the first kind were used as basis functions in developing a Galerkin solution of radiative transport in slab geometry [32, 34–36]. Also, Fourier transforms [36,37] and eigenfunctions expansions [38, 39] have been implemented. It has been observed that an expansion in the optical variable produces fast convergence.

In this study the integro-differential equation of transport in a homogeneous, absorbing and scattering finite slab subjected to prescribed boundary conditions is converted into integral form. The integral version is then solved by expanding the total flux in terms of Chebyshev polynomials in the space variable. To determine the expansion coefficients, the collocation method [33-34] is used to reduce the integral equation into a system of linear algebraic equations to be solved for the expansion coefficients. The knowledge of the expansion coefficients completely determines the total flux, the angular flux and all the physical quantities relevant to the problem.

II. Formulation of the Problem

We consider the radiative energy transfer in a plane parallel absorbing, isotropically scattering and non-emitting slab of optical thickness $2a$. The governing equation is,

$$\mu \frac{\partial I(\tau,\mu)}{\partial \tau} + I(\tau,\mu) = \frac{\omega}{2}\int_{-1}^{+1} I(\tau,\mu')\,d\mu', \qquad -1 \le \mu \le 1 . \tag{1}$$

Here, $I(\tau,\mu)$ is the radiative angular intensity, τ is the optical variable, ω is the single scattering albedo and μ is the direction cosine of propagation of the intensity. The boundary conditions associated with Eq. (1), which describe known externally incident distributions on the slab boundaries, are

$$I(-a, \mu) = F_1(\mu), \qquad \mu > 0 \tag{2a}$$

$$I(a, -\mu) = F_2(\mu), \qquad \mu > 0 \ . \tag{2b}$$

At optical depth τ inside the slab the forward and backward angular intensities are, $\mu > 0$,

$$I(\tau, \mu) = F_1(\mu) \exp[-(a+\tau)/\mu] + \frac{\omega}{2\mu} \int_{-a}^{\tau} \exp[-(\tau - \tau')/\mu] \ I_{\circ}(\tau')\, d\tau' \tag{3}$$

and

$$I(\tau, -\mu) = F_2(\mu) \exp[-(a-\tau)/\mu] + \frac{\omega}{2\mu} \int_{\tau}^{a} \exp[-(\tau' - \tau)/\mu]\, I_{\circ}(\tau')\, d\tau' \ . \tag{4}$$

The exit angular intensities are

$$I(a, \mu) = F_1(\mu) \exp(-2a/\mu) + \frac{\omega}{2\mu} \int_{-a}^{a} \exp[-(a-\tau)/\mu] \ I_{\circ}(\tau)\, d\tau \, , \tag{5a}$$

and

$$I(-a, -\mu) = F_2(\mu) \exp(-2a/\mu) + \frac{\omega}{2\mu} \int_{-a}^{a} \exp[-(a+\tau)/\mu]\, I_{\circ}(\tau)\, d\tau \ . \tag{5b}$$

Other quantities of physical interest are the slab reflectivity and transmissivity defined, respectively, by

$$R = \frac{1}{2\pi} \left\{ \int_0^1 \mu \ [F_1(\mu) + F_2(\mu)] \ d\mu \right\}^{-1} q^{-} \ (-a) \tag{6}$$

and

$$T = \frac{1}{2\pi} \left\{ \int_0^1 \mu \ [F_1(\mu) + F_2(\mu)] \ d\mu \right\}^{-1} q^{+} \ (a) \tag{7}$$

Where q^- and q^+ are the partial heat fluxes at the slab boundaries . In the above equations, the unknown $I_{\circ}(\tau)$ is a solution of the inhomogeneous integral equation,

$$I_{\circ}(\tau) = S(\tau) + \frac{\omega}{2} \int_{-a}^{a} E_1(|\tau - \tau'|)\, I_{\circ}(\tau')\, d\tau', \tag{8}$$

where the inhomogeneous term $S(\tau)$ is defined by

$$S(\tau) = \int_0^1 F_1(\mu) \ \exp[-(a+\tau)/\mu]\, d\mu + \int_0^1 F_2(\mu) \ \exp[-(a-\tau)/\mu]\, d\mu, \tag{9}$$

and $E_n(x)$ denotes the exponential integral function of order n.

III. Method of Solution

To solve Eq. (8) we approximate $I_{\circ}(\tau)$ by the form

$$I_\circ(\tau) = \frac{1}{2} A_0 + \sum_{j=1}^{N}{}^{\backslash} A_j \, T_j(\tau / a), \tag{10}$$

where $T_j(x)$ are Chebyshev polynomials and $\sum^{\backslash}$ means that the last coefficient in the expansion must be halved. Now, insert Eq. (10) into the right hand side of Eq. (8) to get the more accurate approximation

$$I_\circ(\tau) = S(\tau) + \frac{\omega}{4} A_0 \, Q_0(\tau) + \frac{\omega}{2} \sum_{j=1}^{N}{}^{\backslash} A_j \, Q_j(\tau) \tag{11}$$

where we have defined

$$Q_0(\tau) = \int_{-a}^{a} E_1(|\tau - \tau'|) \, d\tau' \quad , \tag{12}$$

$$Q_j(\tau) = H_j(\tau) + (-1)^j H_j(-\tau) \tag{13}$$

and

$$H_j(\tau) = \int_0^{a+\tau} E_1(x) \, T_j\left(\frac{\tau - x}{a}\right) dx. \tag{14}$$

According to the point collocation method, the expansion coefficients in Eq. (10) are solution to the system of algebraic equations

$$\left[\frac{1}{2} - \frac{\omega}{4} Q_0(ax_i)\right] A_0 + \sum_{j=1}^{N}{}^{\backslash} \left[T_j(x_i) - \frac{\omega}{2} Q_j(ax_i)\right] A_j = S(ax_i), \tag{15}$$

where x_i, $i = 0, 1, 2, \ldots\ldots, N$, are the extreme points of $T_j(x)$ [41].

Once the coefficients A_0 and A_j are determined the angular intensity inside the medium at optical depth τ in the positive and negative directions are given respectively by, $\mu > 0$,

$$I(\tau,+\mu) = F_1(\mu) \exp[-(\mathrm{a} + \tau)/\mu] + \frac{\omega}{4} \left[1 - \exp\left[-(\mathrm{a} + \tau)/\mu\right]\right] A_0$$
$$+ \frac{\omega \, a}{2\mu} \exp(-\tau/\mu) \sum_{j=1}^{N}{}^{\backslash} A_j \eta_j(\tau, \mu) \tag{16}$$

and

$$I(\tau,-\mu) = F_2(\mu) \exp[-(\mathrm{a} - \tau)/\mu] + \frac{\omega}{4} \left[1 - \exp[-(\mathrm{a} - \tau)/\mu]\right] A_0$$
$$+ \frac{\omega \, a}{2\mu} \exp(\tau / \mathrm{a}) \sum_{j=1}^{N}{}^{\backslash} (-1)^j A_j \eta_j(-\tau, \mu) \ . \tag{17}$$

where we have defined

$$\eta_j(\tau, \mu) = \frac{\mu}{2a} \sum_{k=0}^{[j/2]} (-1)^{j+k} \frac{j(j-k-1)!}{k!(j-2k)!} \left(\frac{2\mu}{a}\right)^{j-2k}$$
$$\times \left\{\gamma(j - 2k + 1; a/\mu) - \gamma(j - 2k + 1; -\tau/\mu)\right\} . \tag{18}$$

For $I(a,+\mu)$ and $I(-a,-\mu)$ one has

$$I(a,+\mu)=F_1(\mu)\exp(-2a/\mu)+\frac{\omega}{4}\left[1-\exp(-2a/\mu)\right]A_0 + \frac{\omega a}{2\mu}\exp(-a/\mu)\sum_{j=1}^{N}{}' A_j\eta_j(a,\mu) \qquad (19)$$

and

$$I(-a,-\mu)=F_2(\mu)\exp(-2a/\mu)+\frac{\omega}{4}\left[1-\exp(-2a/\mu)\right]A_0 + \frac{\omega a}{2\mu}\exp(-a/\mu)\sum_{j=1}^{N}{}' (-1)^j A_j\eta_j(a,\mu)\ . \qquad (20)$$

By integrating Eqs. (19) and (20) over $\mu\in[0,1]$, the partial total intensities $I^+(a)$ and $I^-(-a)$ are

$$I^+(a)=\int_0^1 F_1(\mu)\exp(-2a/\mu)\,d\mu+\frac{\omega}{4}\left[1-E_2(2a)\right]A_0+\frac{\omega a}{2}\sum_{j=1}^{N}{}' A_j\,\tilde{\eta}_j(a) \qquad (21)$$

and

$$I^-(-a)=\int_0^1 F_2(\mu)\exp(-2a/\mu)\,d\mu+\frac{\omega}{4}\left[1-E_2(2a)\right]A_0+\frac{\omega a}{2}\sum_{j=1}^{N}{}' (-1)^j A_j\,\tilde{\eta}_j(a), \qquad (22)$$

where $\tilde{\eta}_j$ is given by

$$\tilde{\eta}_j(a)=\sum_{k=0}^{[j/2]}\sum_{r=0}^{j-2k}(-1)^{k+r}\frac{j(j-k-1)!}{k!r!(j-2k-r)!}\,2^{j-2k-1}\frac{1}{a^{r+1}}U_r(2a) \qquad (23)$$

and

$$U_r(z)=\frac{z^{r+1}}{r+1}\left\{E_1(z)+\frac{\gamma(r+1,z)}{z^{r+1}}\right\}\ . \qquad (24)$$

Finally to determine the reflectivity and transmissivity of the slab, q^- and q^+ should be obtained. This can be done by multiplying equations (19) and (20) by μ followed by an integration ,

$$\frac{1}{2\pi}q^+(a)=\int_0^1\mu F_1(\mu)\exp(-2a/\mu)\,d\mu+\frac{\omega}{4}\left[\frac{1}{2}-E_3(2a)\right]A_0+\frac{\omega a}{2}\sum_{j=1}^{N}{}' A_j\zeta_j(a) \qquad (25)$$

and

$$\frac{1}{2\pi}q^-(-a)=\int_0^1\mu F_2(\mu)\exp(-2a/\mu)\,d\mu+\frac{\omega}{4}\left[\frac{1}{2}-E_3(2a)\right]A_0+\frac{\omega a}{2}\sum_{j=1}^{N}{}' (-1)^j A_j\,\xi_j(a), \qquad (26)$$

where ξ_j is given by

$$\xi_j(a)=\exp(-a)\,\eta_j(a,\mu=1) - a\sum_{k=0}^{[j/2]}\sum_{r=0}^{j-2k}(-1)^{k+r}\frac{j(j-k-1)!}{k!r!(j-2k-r)!}\cdot\frac{2^{j-2k-1}}{a^{r+2}}U_{r+1}(2a), \qquad (27)$$

IV. Numerical Results and Discussion

In this section, we present the numerical results for the proposed method of solution in the study of the transport properties of a semitransparent planar slab with isotropic scattering. For the sake of numerical comparison the externally incident radiation at the right boundary $F_2(\mu)=0$. At the inlet, $\tau=-a$, isotropic and normal incidence , each of unit strength , are assumed. To obtain the final solutions, we need to evaluate the expansion coefficients. They are determined by solving a linear system of algebraic equations. Thus, the intensity of radiation and the net flux can be known everywhere in the medium. However, for the purpose of comparison, numerical calculations are done for the transmissivity and reflectivity of slabs with various values of ω, which is the single scattering albedo. The obtained results are tabulated in Tables 1 to 4 for the cases of isotropic and normal incidence. The results for the case of isotropic incidence are compared with the values of the discrete ordinate method [18] and the exact values of Lii, [42] for $\omega < 1$, and Busbridge[43] for $\omega = 1$. The results for the case of normal incidence are compared with the approximate values calculated with Pomraning-Eddington variational method and Case's eigenvalue method reported in [44] and with those of discrete ordinate method [18] . From Tables 1 to 4, it can be seen that the results calculated by the proposed method agree very well with the exact and discrete ordinate results.

Numerical results are also performed for the transmitted and reflected angular intensities at the boundary of a slab with different optical thickness and selected values of ω. In all the calculations the largest approximation order is N=10. In Tables 5 and 6, we list numerical values for the transmitted and reflected angular intensities of a slab with three values of the optical thickness at selected values of ω in the case of isotropic incidence, and the results are compared with those which are obtained by the method of Ref. [45] where the Legendre polynomial expansion is used, up to the same approximation order N=10. In Tables 7 and 8, the listed values are same like those of Tables 5 and 6 with the exception that the case of normal incidence is considered for two values of the optical thickness. For normal incidence, our results of the transmitted and reflected intensities are compared with the results obtained by Chandrasekhar X- and Y- functions method [1] . In general, comparison of our results with the available data shows good agreement.

Table 1. The transmissivity of slabs at various values of ω in case of isotropic incidence

ω	N					
	1	3	7	10	DOM[18]	Exact [42,43]
	2a = 0.5					
0.1	0.46064	0.45799	0.45801	0.45797	0.45715	0.4578
0.2	0.47942	0.47429	0.47432	0.47426	0.47332	0.4744
0.3	0.49969	0.49233	0.49239	0.49230	0.49122	0.4925
0.4	0.52164	0.51242	0.51249	0.51238	0.51117	0.5125
0.5	0.54548	0.53490	0.53499	0.53488	0.53351	0.5350
0.6	0.57143	0.56022	0.56034	0.56022	0.55869	0.5603
0.7	0.59979	0.58895	0.58909	0.58899	0.58728	0.5891
0.8	0.63090	0.62180	0.62197	0.62189	0.61998	0.6220
0.9	0.66516	0.65972	0.65992	0.65987	0.65775	0.6599
1.0	0.70305	0.70394	0.70417	0.70416		
	2a = 1					
0.1	0.23780	0.23180	0.23185	0.23185	0.23418	0.2317
0.2	0.25820	0.24616	0.24627	0.24627	0.24838	0.2459
0.3	0.28090	0.26292	0.26309	0.26310	0.26496	0.2627
0.4	0.30627	0.28271	0.28296	0.28296	0.28453	0.2825
0.5	0.33476	0.30638	0.30671	0.30671	0.30796	0.3063
0.6	0.36694	0.33513	0.33554	0.33555	0.33642	0.3352
0.7	0.40350	0.37069	0.37119	0.37120	0.37166	0.3710
0.8	0.44532	0.41563	0.41624	0.41624	0.41625	0.4162
0.9	0.49354	0.47401	0.47474	0.47475	0.47430	0.4748
1.0	0.54962	0.55253	0.55340	0.55340		0.5534
	2a = 2					
0.1	0.07489	0.06581	0.06585	0.06586	0.06611	0.0658
0.2	0.09157	0.07262	0.07274	0.07274	0.07304	0.0728
0.3	0.11073	0.08116	0.08138	0.08138	0.08171	0.0814
0.4	0.13284	0.09209	0.09245	0.09246	0.09282	0.0925
0.5	0.15856	0.10650	0.10705	0.10706	0.10742	0.1071
0.6	0.18869	0.12612	0.12690	0.12691	0.12725	0.1269
0.7	0.22432	0.15400	0.15506	0.15507	0.15535	0.1551
0.8	0.26685	0.19585	0.19725	0.19727	0.19742	0.1973
0.9	0.31821	0.26370	0.26556	0.26558	0.26552	0.2656
1.0	0.38108	0.38740	0.39003	0.39006		0.3901
	2a = 5					
0.1	0.00891	0.00267	0.00203	0.00205		0.0020
0.2	0.01731	0.00365	0.00242	0.00245		0.0024
0.3	0.02725	0.00474	0.00299	0.00303		0.0030
0.4	0.03910	0.00601	0.00385	0.00390		0.0039
0.5	0.05335	0.00761	0.00524	0.00529		0.0053
0.6	0.07065	0.00996	0.00766	0.00772		0.0077
0.7	0.09188	0.01414	0.01233	0.01240		0.0124
0.8	0.11827	0.02356	0.02286	0.02293		0.0229
0.9	0.15157	0.05209	0.05335	0.05342		0.0534
1.0	0.19429	0.20076	0.20749	0.20763		0.2077

Table 2. The reflectivity of slabs at various values of ω in case of isotropic incidence

ω	N					
	1	3	7	10	DOM [18]	Exact [42, 43]
	2a = 0.5					
0.1	0.02079	0.01782	0.01777	0.01774	0.01803	0.0178
0.2	0.04299	0.03725	0.03715	0.03709	0.03767	0.0372
0.3	0.06676	0.05852	0.05839	0.05830	0.05916	0.0584
0.4	0.09229	0.08194	0.08178	0.08167	0.08279	0.0818
0.5	0.11978	0.10787	0.10768	0.10757	0.10894	0.1077
0.6	0.14947	0.13676	0.13655	0.13643	0.13804	0.1365
0.7	0.18166	0.16918	0.16895	0.16885	0.17068	0.1690
0.8	0.21669	0.20585	0.20562	0.20554	0.20757	0.2056
0.9	0.25496	0.24773	0.24750	0.24745	0.24966	0.2475
1.0	0.29695	0.29606	0.29583	0.29584		
	2a = 1					
0.1	0.02840	0.02101	0.02075	0.02075	0.02092	0.0207
0.2	0.05918	0.04443	0.04394	0.04394	0.04425	0.0439
0.3	0.09265	0.07078	0.07008	0.07007	0.07051	0.0701
0.4	0.12922	0.10072	0.09985	0.09985	0.10038	0.0999
0.5	0.16937	0.13517	0.13418	0.13417	0.13477	0.1342
0.6	0.21368	0.17540	0.17432	0.17431	0.17495	0.1743
0.7	0.26289	0.22321	0.22208	0.22207	0.22272	0.2221
0.8	0.31791	0.28127	0.28016	0.28015	0.28080	0.2806
0.9	0.37991	0.35375	0.35272	0.35271	0.35338	0.3527
1.0	0.45038	0.44747	0.44660	0.44660	0.44740	0.4466
	2a = 2					
0.1	0.03659	0.02285	0.02164	0.02163	0.02181	0.0216
0.2	0.07658	0.04840	0.04610	0.04608	0.04642	0.0461
0.3	0.12047	0.07735	0.07410	0.07406	0.07455	0.0741
0.4	0.16891	0.11067	0.10662	0.10658	0.10717	0.1066
0.5	0.22269	0.14980	0.14515	0.14510	0.14576	0.1451
0.6	0.28282	0.19696	0.19193	0.19188	0.19257	0.1919
0.7	0.35055	0.25580	0.25068	0.25063	0.25128	0.2506
0.8	0.42755	0.33287	0.32801	0.32796	0.32851	0.3280
0.9	0.51602	0.44132	0.43720	0.43716	0.43753	0.4376
1.0	0.61892	0.61260	0.60997	0.60994	0.61020	0.6099
	2a = 5					
0.1	0.04544	0.02792	0.02187	0.02172		0.0217
0.2	0.09546	0.05848	0.04660	0.04632		0.0463
0.3	0.15081	0.09229	0.07492	0.07452		0.0745
0.4	0.21240	0.13020	0.10790	0.10742		0.1073
0.5	0.28137	0.17354	0.14719	0.14664		0.1465
0.6	0.35917	0.22450	0.19539	0.19481		0.1947
0.7	0.44764	0.28715	0.25721	0.25663		0.2565
0.8	0.54920	0.37026	0.34232	0.34178		0.3417
0.9	0.66708	0.49854	0.47684	0.47642		0.4763
1.0	0.80571	0.79924	0.79251	0.79237	0.79240	0.7923

Table 3. The transmissivity of slabs at various values of ω in case of normal incidence

ω	N					
	1	3	7	10	DOM[18]	Vart. [44]
	2a = 0.5					
0.1	0.61761	0.61751	0.61751	0.61751	0.61749	0.61751
0.2	0.62952	0.62957	0.62957	0.62958	0.62954	0.62958
0.3	0.64234	0.64290	0.64290	0.64292	0.64284	0.64292
0.4	0.65619	0.65770	0.65770	0.65773	0.65762	0.65774
0.5	0.67120	0.67422	0.67423	0.67427	0.67412	0.67428
0.6	0.68750	0.69278	0.69280	0.69286	0.69267	0.69286
0.7	0.70527	0.71379	0.71381	0.71391	0.71365	0.71387
0.8	0.72472	0.73776	0.73778	0.73792	0.73594	0.73784
0.9	0.74609	0.76536	0.76538	0.76556	0.76516	0.76542
1.0	0.76967	0.79746	0.79749	0.79773		
	2a = 1					
0.1	0.37997	0.37923	0.37923	0.37923	0.37916	0.37924
0.2	0.39328	0.39225	0.39226	0.39226	0.39211	0.39231
0.3	0.40801	0.40735	0.40736	0.40736	0.40712	0.40746
0.4	0.42437	0.42504	0.42505	0.42505	0.42471	0.42523
0.5	0.44264	0.44603	0.44606	0.44606	0.44561	0.44631
0.6	0.46317	0.47134	0.47137	0.47137	0.47082	0.47169
0.7	0.48636	0.50239	0.50243	0.50244	0.50179	0.50279
0.8	0.51274	0.54134	0.54140	0.54140	0.54070	0.54173
0.9	0.54300	0.59155	0.59162	0.59162	0.59097	0.59185
1.0	0.57801	0.65857	0.65867	0.65867	0.65830	

	2a = 2					
0.1	0.14441	0.14207	0.14208	0.14208	0.14209	0.14211
0.2	0.15468	0.15023	0.15024	0.15024	0.15025	0.15038
0.3	0.16637	0.16027	0.16030	0.16030	0.16031	0.16062
0.4	0.17976	0.17292	0.17296	0.17296	0.17296	0.17351
0.5	0.19521	0.18926	0.18932	0.18932	0.18929	0.19011
0.6	0.21316	0.21106	0.21114	0.21114	0.21107	0.21215
0.7	0.23423	0.24138	0.24145	0.24145	0.24132	0.24261
0.8	0.25918	0.28590	0.28594	0.28595	0.28573	0.28709
0.9	0.28910	0.35649	0.35649	0.35650	0.35622	0.35732
1.0	0.32546	0.48239	0.48246	0.48248	0.48240	
	2a = 5					
0.1	0.01044	0.00746	0.00732	0.00732		0.00734
0.2	0.01479	0.00833	0.00808	0.00808		0.00817
0.3	0.01992	0.00943	0.00913	0.00913		0.00934
0.4	0.02602	0.01092	0.01064	0.01063		0.01100
0.5	0.03334	0.01310	0.01293	0.01292		0.01346
0.6	0.04221	0.01663	0.01667	0.01666		0.01738
0.7	0.05308	0.02311	0.02347	0.02344		0.02434
0.8	0.06657	0.03710	0.03780	0.03776		0.03880
0.9	0.08356	0.07588	0.07667	0.07664		0.07762
1.0	0.10532	0.25870	0.26121	0.26123	0.26140	

Table 4. The reflectivity of slabs at various values of ω in case of normal incidence

ω	N					
	1	3	7	10	DOM[18]	Vart. [44]
	2a = 0.5					
0.1	0.01206	0.01196	0.01196	0.01196	0.01200	0.01196
0.2	0.02497	0.02503	0.02504	0.02504	0.02513	0.02505
0.3	0.03881	0.03941	0.03942	0.03943	0.03954	0.03944
0.4	0.05371	0.05528	0.05530	0.05533	0.05547	0.05534
0.5	0.06978	0.07292	0.07295	0.07299	0.07315	0.07300
0.6	0.08716	0.09264	0.09268	0.09275	0.09291	0.09275
0.7	0.10607	0.11485	0.11491	0.11500	0.11515	0.11500
0.8	0.12666	0.14006	0.14013	0.14027	0.14039	0.14020
0.9	0.14920	0.16894	0.16904	0.16922	0.16930	0.16909
1.0	0.17399	0.20238	0.20250	0.20274		
	2a = 1					
0.1	0.01579	0.01503	0.01504	0.01504	0.01504	0.01505
0.2	0.03296	0.03195	0.03198	0.03198	0.03198	0.03204
0.3	0.05168	0.05118	0.05125	0.05125	0.05122	0.05137
0.4	0.07220	0.07325	0.07338	0.07338	0.07333	0.07358
0.5	0.09481	0.09892	0.09912	0.09912	0.09904	0.09941
0.6	0.11983	0.12921	0.12950	0.12951	0.12941	0.12987
0.7	0.14772	0.16560	0.16601	0.16602	0.16591	0.16643
0.8	0.17900	0.21028	0.21084	0.21085	0.21071	0.21125
0.9	0.21438	0.26667	0.26740	0.26741	0.26746	0.26771
1.0	0.25473	0.34037	0.34132	0.34133	0.34170	
	2a = 2					
0.1	0.01895	0.01622	0.01625	0.01625	0.01626	0.01629
0.2	0.03968	0.03472	0.03487	0.03487	0.03489	0.03506
0.3	0.06248	0.05613	0.05649	0.05650	0.05651	0.05693
0.4	0.08770	0.08135	0.08203	0.08204	0.08203	0.08278
0.5	0.11575	0.11169	0.11282	0.11283	0.11279	0.11392
0.6	0.14718	0.14922	0.15096	0.15097	0.15088	0.15237
0.7	0.18267	0.19739	0.19990	0.19993	0.19977	0.20154
0.8	0.22311	0.26240	0.26590	0.26593	0.26572	0.26785
0.9	0.26969	0.35684	0.36159	0.36164	0.36144	0.36297
1.0	0.32403	0.51093	0.51743	0.51750	0.51760	

	2a = 5					
0.1	0.02274	0.01661	0.01639	0.01639		0.01648
0.2	0.04777	0.03520	0.03521	0.03523		0.03561
0.3	0.07548	0.05633	0.05717	0.05721		0.05808
0.4	0.10631	0.08078	0.08327	0.08334		0.08485
0.5	0.14083	0.10986	0.11508	0.11520		0.11739
0.6	0.17978	0.14573	0.15518	0.15536		0.15822
0.7	0.22408	0.19254	0.20831	0.20856		0.21198
0.8	0.27493	0.25951	0.28452	0.28488		0.28886
0.9	0.33397	0.37337	0.41193	0.41244		0.41595
1.0	0.40340	0.67629	0.73778	0.73856	0.73860	

Table 5. The reflected angular intensity in case of isotropic incidence
(i) Present results (ii) Method of Ref. [45]

ω	μ	2a = 0.1		2a = 1		2a = 5	
		(i)	(ii)	(i)	(ii)	(i)	(ii)
0.2	0.1	0.05607	0.05607	0.08246	0.08245	0.08339	0.08255
	0.2	0.03448	0.03448	0.07042	0.07042	0.07099	0.07075
	0.3	0.02474	0.02473	0.06162	0.06162	0.06247	0.06237
	0.4	0.01926	0.01926	0.05467	0.05467	0.05602	0.05597
	0.5	0.01576	0.01576	0.04903	0.04903	0.05089	0.05086
	0.6	0.01334	0.01334	0.04438	0.04438	0.04668	0.04665
	0.7	0.01156	0.01156	0.04050	0.04050	0.04314	0.04312
	0.8	0.01020	0.01020	0.03721	0.03721	0.04013	0.04011
	0.9	0.00912	0.00912	0.03441	0.03441	0.03752	0.03751
	1.0	0.00825	0.00825	0.03198	0.03198	0.03524	0.03523
0.3	0.1	0.08552	0.08552	0.12939	0.12938	0.13100	0.12966
	0.2	0.05259	0.05260	0.11130	0.11129	0.11250	0.11212
	0.3	0.03773	0.03773	0.09782	0.09782	0.09958	0.09943
	0.4	0.02938	0.02938	0.08703	0.08703	0.08969	0.08962
	0.5	0.02405	0.02405	0.07821	0.07820	0.08176	0.08172
	0.6	0.02035	0.02035	0.07089	0.07089	0.07521	0.07518
	0.7	0.01764	0.01764	0.06476	0.06476	0.06968	0.06966
	0.8	0.01556	0.01556	0.05956	0.05955	0.06495	0.06493
	0.9	0.01392	0.01392	0.05510	0.05510	0.06083	0.06082
	1.0	0.01259	0.01260	0.05125	0.05125	0.05723	0.05721
0.7	0.1	0.21391	0.21391	0.37817	0.37815	0.39189	0.38900
	0.2	0.13165	0.13165	0.33708	0.33708	0.35278	0.35197
	0.3	0.09448	0.09448	0.30279	0.30279	0.32297	0.32271
	0.4	0.07358	0.07358	0.27322	0.27322	0.29866	0.29857
	0.5	0.06023	0.06023	0.24788	0.24788	0.27818	0.27814
	0.6	0.05097	0.0597	0.22625	0.22625	0.26055	0.26053
	0.7	0.04417	0.04417	0.20774	0.20774	0.24517	0.24516
	0.8	0.03897	0.03897	0.19183	0.19182	0.23159	0.23158
	0.9	0.03487	0.03487	0.17804	0.17804	0.21949	0.21949
	1.0	0.03154	0.03154	0.16602	0.16602	0.20863	0.20863
0.9	0.1	0.28532	0.28532	0.56900	0.56899	0.62968	0.62743
	0.2	0.17566	0.17566	0.51941	0.51940	0.59106	0.59045
	0.3	0.12608	0.12608	0.47341	0.47341	0.55912	0.55895
	0.4	0.09820	0.09820	0.43117	0.43117	0.53136	0.53131
	0.5	0.08038	0.08038	0.39366	0.39367	0.50666	0.50664
	0.6	0.06802	0.06802	0.36093	0.36093	0.48438	0.48439
	0.7	0.05895	0.05895	0.33252	0.33252	0.46411	0.46412
	0.8	0.05201	0.05201	0.30783	0.30783	0.44554	0.44554
	0.9	0.04654	0.04654	0.28630	0.28630	0.42841	0.42842
	1.0	0.04210	0.04210	0.26741	0.26741	0.41256	0.41256
1.0	0.1	0.32308	0.32307	0.69767	0.69767	0.88819	0.88727
	0.2	0.19894	0.19894	0.64597	0.64597	0.86969	0.86946
	0.3	0.14280	0.14280	0.59387	0.59387	0.85234	0.85229
	0.4	0.11122	0.11122	0.54386	0.54386	0.83553	0.83552
	0.5	0.09104	0.09104	0.49838	0.49838	0.81901	0.81902
	0.6	0.07705	0.07705	0.45813	0.45814	0.80269	0.80270
	0.7	0.06678	0.06678	0.42289	0.42289	0.78650	0.78651
	0.8	0.05892	0.05891	0.39208	0.39209	0.77044	0.77045
	0.9	0.05271	0.05271	0.36509	0.36509	0.75450	0.75451
	1.0	0.04769	0.04769	0.34133	0.34133	0.73873	0.73874

Table 6. The transmitted angular intensity in case of isotropic incidence
(i) Present results (ii) Method of Ref. [45]

ω	μ	2a = 0.1		2a = 1		2a = 5	
		(i)	(ii)	(i)	(ii)	(i)	(ii)
0.2	0.1	0.42123	0.42123	0.02089	0.02089	0.00017	0.00024
	0.2	0.64015	0.64015	0.03160	0.03160	0.00021	0.00021
	0.3	0.74085	0.74085	0.06345	0.06345	0.00024	0.00024
	0.4	0.79782	0.79782	0.11111	0.11111	0.00030	0.00029
	0.5	0.83434	0.83434	0.16445	0.16444	0.00042	0.00041
	0.6	0.85971	0.85971	0.21741	0.21741	0.00073	0.00072
	0.7	0.87835	0.87845	0.26728	0.26728	0.00143	0.00143
	0.8	0.89263	0.89263	0.31308	0.31308	0.00278	0.00277
	0.9	0.90391	0.90391	0.35466	0.35467	0.00495	0.00495
	1.0	0.91305	0.91305	0.39226	0.39226	0.00808	0.00808
0.3	0.1	0.44930	0.44930	0.03447	0.03446	0.00033	0.00059
	0.2	0.65783	0.65783	0.04765	0.04765	0.00041	0.00046
	0.3	0.75364	0.75364	0.08117	0.08117	0.00047	0.00048
	0.4	0.80782	0.80782	0.12946	0.12946	0.00057	0.00057
	0.5	0.84254	0.84254	0.18273	0.18273	0.00076	0.00075
	0.6	0.86666	0.86666	0.23526	0.23526	0.00116	0.00115
	0.7	0.88439	0.88439	0.28450	0.28450	0.00198	0.00198
	0.8	0.89796	0.89796	0.32960	0.32960	0.00347	0.00347
	0.9	0.90868	0.90868	0.37047	0.37047	0.00581	0.00581
	1.0	0.91737	0.91737	0.40736	0.40736	0.00913	0.00913
0.7	0.1	0.57208	0.57208	0.12856	0.12856	0.00401	0.00530
	0.2	0.73512	0.73512	0.15687	0.15687	0.00472	0.00503
	0.3	0.80953	0.80954	0.19902	0.19902	0.00548	0.00553
	0.4	0.85152	0.85152	0.24945	0.24945	0.00639	0.00638
	0.5	0.87839	0.87839	0.30092	0.30092	0.00756	0.00753
	0.6	0.89705	0.89705	0.34964	0.34964	0.00915	0.00912
	0.7	0.91075	0.91075	0.39419	0.39419	0.01137	0.01134
	0.8	0.92124	0.92124	0.43433	0.43433	0.01441	0.01439
	0.9	0.92952	0.92952	0.47027	0.47027	0.01842	0.01839
	1.0	0.93816	0.93623	0.50244	0.50244	0.02345	0.02343
0.9	0.1	0.28532	0.64062	0.22575	0.22575	0.02466	0.02581
	0.2	0.17566	0.77823	0.26760	0.26760	0.02884	0.02910
	0.3	0.12608	0.84070	0.31588	0.31588	0.03303	0.03306
	0.4	0.09820	0.87588	0.36648	0.36648	0.03744	0.03742
	0.5	0.08038	0.89838	0.41484	0.41484	0.04225	0.04221
	0.6	0.06802	0.91399	0.45894	0.45894	0.04759	0.04755
	0.7	0.05895	0.92545	0.49834	0.49834	0.05362	0.05358
	0.8	0.05201	0.93421	0.53327	0.53327	0.06045	0.06042
	0.9	0.04654	0.94113	0.56420	0.56420	0.06813	0.06810
	1.0	0.04210	0.94675	0.59163	0.59163	0.07665	0.07663
1.0	0.1	0.67692	0.67693	0.30233	0.30234	0.11181	0.11273
	0.2	0.80106	0.80106	0.35403	0.35403	0.13031	0.13054
	0.3	0.85720	0.85720	0.40613	0.40613	0.14766	0.14771
	0.4	0.88878	0.88878	0.45614	0.45615	0.16447	0.16448
	0.5	0.90896	0.90896	0.50162	0.50162	0.18099	0.18098
	0.6	0.92295	0.92295	0.54187	0.54187	0.19731	0.19730
	0.7	0.93322	0.93323	0.57711	0.57711	0.21350	0.21349
	0.8	0.94108	0.94108	0.60792	0.60792	0.22956	0.22955
	0.9	0.94729	0.94729	0.63491	0.63491	0.24550	0.24549
	1.0	0.95231	0.95231	0.65867	0.65867	0.26127	0.26127

Table 7. The reflected angular intensity in case of normal incidence
(i) Present results (ii) Chandrasekhar method [1]

ω	μ	2a = 1		2a = 5	
		(i)	(ii)	(i)	(ii)
0.2	0.1	0.10000	0.10000	0.10048	0.10055
	0.2	0.09244	0.09244	0.09336	0.09338
	0.3	0.08495	0.08495	0.08697	0.08698
	0.4	0.07781	0.07781	0.08131	0.08132
	0.5	0.07134	0.07134	0.07631	0.07631
	0.6	0.06561	0.06561	0.07185	0.07186
	0.7	0.06059	0.06059	0.06788	0.06788
	0.8	0.05619	0.05619	0.06431	0.06431
	0.9	0.05234	0.05234	0.06109	0.06109
	1.0	0.04895	0.04895	0.05817	0.05817
0.3	0.1	0.15817	0.158187	0.15958	0.15979
	0.2	0.14710	0.14711	0.14940	0.14946
	0.3	0.13564	0.13566	0.13992	0.13995
	0.4	0.12451	0.12453	0.13136	0.13137
	0.5	0.11430	0.11433	0.12367	0.12368
	0.6	0.10522	0.10525	0.11677	0.11677
	0.7	0.09724	0.09727	0.11055	0.11056
	0.8	0.09024	0.09027	0.10494	0.10495
	0.9	0.08409	0.08421	0.09985	0.09986
	1.0	0.07866	0.07870	0.09522	0.09522

0.7	0.1	0.48152	0.48154	0.51016	0.51161
	0.2	0.46088	0.46089	0.49780	0.49824
	0.3	0.43209	0.43209	0.48069	0.48087
	0.4	0.40066	0.40067	0.46238	0.46247
	0.5	0.37027	0.37028	0.44415	0.44420
	0.6	0.34243	0.34244	0.42654	0.42657
	0.7	0.31752	0.31753	0.40977	0.40980
	0.8	0.29541	0.29543	0.39392	0.39394
	0.9	0.27584	0.27585	0.37901	0.37903
	1.0	0.25847	0.25848	0.36500	0.36502
0.9	0.1	0.74531	0.74534	0.88190	0.88492
	0.2	0.72660	0.72662	0.89245	0.89338
	0.3	0.68848	0.68849	0.88770	0.88807
	0.4	0.64253	0.64254	0.87566	0.87585
	0.5	0.59630	0.59631	0.85972	0.85984
	0.6	0.55309	0.55310	0.84164	0.84172
	0.7	0.51394	0.51396	0.82243	0.82249
	0.8	0.47894	0.47895	0.80272	0.80277
	0.9	0.44777	0.44778	0.78290	0.78294
	1.0	0.41910	0.42001	0.76324	0.76328
1.0	0.1	0.92972	0.92975	1.36150	1.37281
	0.2	0.91605	0.91607	1.42750	1.43041
	0.3	0.87331	0.87333	1.46468	1.46538
	0.4	0.81805	0.81807	1.48544	1.48553
	0.5	0.76102	0.76104	1.49524	1.49516
	0.6	0.70705	0.70707	1.49709	1.49697
	0.7	0.65782	0.65783	1.49293	1.49280
	0.8	0.61358	0.61359	1.48410	1.48398
	0.9	0.57406	0.57408	1.47161	1.47150
	1.0	0.53877	0.53873	1.45629	1.45618

Table 8. The transmitted angular intensity in case of normal incidence
(i) Present results (ii) Chandrasekhar method [1]

ω	μ	2a = 1		2a = 5	
		(i)	(ii)	(i)	(ii)
0.2	0.1	0.04718	0.04718	0.00101	0.00103
	0.2	0.05219	0.05220	0.00114	0.00115
	0.3	0.05476	0.05476	0.00131	0.00132
	0.4	0.05485	0.05485	0.00153	0.00153
	0.5	0.05346	0.05346	0.00182	0.00182
	0.6	0.05137	0.05137	0.00219	0.00219
	0.7	0.04902	0.04902	0.00265	0.00265
	0.8	0.04662	0.04662	0.00318	0.00318
	0.9	0.04430	0.04430	0.00376	0.00376
	1.0	0.40998	0.40998	0.01111	0.01111
0.3	0.1	0.07665	0.07659	0.00181	0.00183
	0.2	0.08486	0.08479	0.00207	0.00208
	0.3	0.08893	0.08887	0.00238	0.00239
	0.4	0.08896	0.08890	0.00278	0.00278
	0.5	0.08663	0.08658	0.00329	0.00329
	0.6	0.08319	0.08314	0.00394	0.00394
	0.7	0.07933	0.07928	0.00473	0.00473
	0.8	0.07542	0.07537	0.00564	0.00564
	0.9	0.07164	0.07160	0.00663	0.00663
	1.0	0.43595	0.43594	0.01441	0.01441
0.7	0.1	0.26602	0.26601	0.01438	0.01441
	0.2	0.29495	0.29494	0.01669	0.01672
	0.3	0.30721	0.30720	0.01924	0.01925
	0.4	0.30552	0.30552	0.02214	0.02215
	0.5	0.29616	0.29615	0.02553	0.02554
	0.6	0.28340	0.28339	0.02944	0.02945
	0.7	0.26953	0.26953	0.03383	0.03384
	0.8	0.25571	0.25570	0.03857	0.03857
	0.9	0.24248	0.24247	0.04349	0.04349
	1.0	0.59794	0.59794	0.05518	0.05520
0.9	0.1	0.44805	0.44806	0.06952	0.06972
	0.2	0.49665	0.49665	0.08103	0.08114
	0.3	0.51516	0.51516	0.09253	0.09259
	0.4	0.51040	0.51040	0.10445	0.10449
	0.5	0.49336	0.49335	0.11700	0.11702
	0.6	0.47107	0.47107	0.13015	0.13017
	0.7	0.44726	0.44726	0.14370	0.14371
	0.8	0.42376	0.42377	0.15732	0.15733
	0.9	0.40141	0.40142	0.17067	0.17068
	1.0	0.74841	0.74841	0.19019	0.19028

1.0	0.1	0.58678	0.58680		0.27985	0.27458
	0.2	0.65012	0.65012		0.32584	0.32479
	0.3	0.67272	0.67272		0.36889	0.36895
	0.4	0.66508	0.66508		0.41037	0.41068
	0.5	0.64182	0.64182		0.45064	0.45097
	0.6	0.61208	0.61207		0.48955	0.48985
	0.7	0.58060	0.58059		0.52668	0.52694
	0.8	0.54968	0.54967		0.56153	0.56176
	0.9	0.52037	0.52039		0.59364	0.59384
	1.0	0.86094	0.86094		0.62946	0.62964

V. Conclusion

A spatial Chebyshev polynomial expansion, in connection with collocation method, is applied to the radiative transfer problem in one-dimensional absorbing, scattering and non-emitting planar slab subjected to prescribed externally incident radiation. The integral form of the considered problem is formulated and the total intensity is expanded in a series of Chebyshev polynomials of the first kind. The expansion coefficients are solutions to a linear, inhomogeneous system of algebraic equations. Once the expansion coefficients are determined all the physical quantities relevant to the problem could be calculated. The accuracy of the proposed method is validated by performing numerical results for the reflectivity, transmissivity, and angular intensity of a semitransparent planar slab, with isotropic scattering, subjected to isotropic and normal incidence.

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