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A Semi-Analytical Solution to Milne Problem in the Case of Multiple Synthetic Scattering

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Abstract: *The Milne problem is studied for the case of one speed, time independent, plane symmetric transport equation. The scattering law is assumed isotropic plus a backward and forward leak. First of all, the solution for the angular intensity relevant to the assumed scattering law is expressed in terms of the solution of the transport equation with isotropic scattering. The integral version of the isotropic like transport equation is then solved using a suitable set of trial functions. The unknown expansion coefficients in the trial functions are shown to be solutions of a system of linear algebraic equations. Numerical results are listed and compared with the existing results.*
Keywords: *Transport equation, Angular intensity, Milne problem, Approximate expansion.*

I. Introduction

Various scattering models have been currently used in a series of papers to solve problems in finite and semi-infinite media with different techniques [1-15]. Here, our aim is to determine the particle distribution everywhere in an infinite source-free half-space with zero incident flux for which the scattering law is isotropic scattering plus a backward and forward leak. Also, we wish here to extend a method previously introduced [16-18] to demonstrate the computational merits of the method when applied to the anisotropic scattering kernel mentioned above. A scattering law of this kind was used by Inönü [2] which led him to prove a theorem such that the Boltzmann equation with synthetic scattering kernel is connected to the one with isotropic scattering. This connection is made by anisotropy parameters ℓ , backward leak parameter, and m, forward leak parameter, for which the new eigenvalues, eigen-functions and the number of secondaries are described in terms of the old ones [1].

Let us consider one-speed, time independent homogeneous transport equation:

$$\mu \frac{\partial}{\partial z}\psi(z,\mu) + \psi(z,\mu) = \omega_0 \int_{-1}^{1} \Pi(\mu' \to \mu)\, \psi(z,\mu')\, d\mu' \tag{1}$$

where ψ is the angular density, with distance measured in units of mean free path, μ is the direction cosine of the angle between the positive z-axis and the particle velocity vector, and ω_0 is the mean number of secondaries per collision. Here $\Pi(\mu' \to \mu)$ represents the scattering law, which we write as [4]

$$\Pi(\mu', \mu) = \ell\delta(\mu' + \mu) + m\delta(\mu' - \mu) + \frac{1}{2}n, \tag{2}$$

where ℓ and m are real constants in the range $0 \le \ell, m \le 1$ and give the fraction of particles which emerge from a collision in the backward and forward directions, respectively. In addition
n =1- $\ell - m$ and gives the fraction of particles which emerge isotropically from a collision. Using equation (2) we can write equation (1) as

$$\mu \frac{\partial}{\partial z}\psi(z,\mu) + (1 - m\omega_0)\psi(z,\mu) = \ell\omega_0\, \psi(z,-\mu) + \frac{1}{2}n\omega_0 \int_{-1}^{1} \psi(z,\mu')\, d\mu'. \tag{3}$$

In terms of the reduced optical variable

$$\tau = (1 - m\omega_0)z, \tag{4}$$

equation (3) becomes

$$\mu \frac{\partial}{\partial \tau}\psi(\tau,\mu) + \psi(\tau,\mu) = \alpha\psi(\tau,\mu) + \frac{1}{2}\beta \int_{-1}^{1} \psi(\tau,\mu')\, d\mu', \tag{5}$$

where

$$\boldsymbol{\alpha} = \frac{\ell \boldsymbol{\omega_0}}{\mathbf{1 - m\omega_0}} \tag{6}$$

and

$$\boldsymbol{\beta} = \frac{\mathbf{n\omega_0}}{\mathbf{1 - m\omega_0}} \tag{7}$$

Equation (5) shows that forward scattering does not introduce any analytical complications since the form of the resulting equation of transport is no different from the usual equation. However, backward scattering does lead to an equation of transport considerably different from the usual one. In the following section the Milne problem will be considered, which is the problem to obtain the angular intensity everywhere in the half-space $0 \le \tau \le \infty$ with zero incident intensity where particles are diffusing from a source at $\tau = +\infty$. In the region $\tau < 0$ there is a vacuum.

II. The Milne Problem

For this problem, as mentioned previously, we seek solutions of the homogeneous transport equation, Eq. (5), subject to the boundary condition at z = 0,

$$\boldsymbol{\psi(0,\mu) = 0 \ , \qquad \mu > 0} \tag{8}$$

As discussed by Inönü [2], we can reduce the problem defined by Eqs. (5) and (8) to the one for isotropic scattering by writing

$$\boldsymbol{\psi(\tau,\mu)} = \frac{\mathbf{B+1}}{\mathbf{2}} \boldsymbol{\Psi(x,\mu)} - \frac{\mathbf{B-1}}{\mathbf{2}} \boldsymbol{\Psi(x,-\mu)} \tag{9}$$

where

$$\mathbf{x} = \boldsymbol{\tau(1-\alpha^2)^{\frac{1}{2}}}, \tag{10}$$

$$\mathbf{B} = \left(\frac{\mathbf{1-\alpha}}{\mathbf{1+\alpha}}\right)^{\frac{1}{2}} , \tag{11}$$

and $\Psi(x,\mu)$ is a solution to

$$\boldsymbol{\mu}\frac{\partial}{\partial \mathbf{x}}\boldsymbol{\Psi(x,\mu) + \Psi(x,\mu)} = \frac{\boldsymbol{\omega}}{\mathbf{2}} \int_{-1}^{1} \boldsymbol{\Psi(x,\mu')d\mu'} \tag{12}$$

together with the definition

$$\boldsymbol{\omega} = \frac{\boldsymbol{\beta}}{\mathbf{1-\alpha}}. \tag{13}$$

Using the boundary condition in Eq. (8), we see that Eq. (9) at $\tau = x = 0$ assumes the form

$$\Psi(0,\mu) = R\Psi(0,-\mu), \quad \mu > 0 \tag{14}$$

which expresses the new boundary condition at $x = 0$, where

$$\mathbf{R} = \frac{\mathbf{B-1}}{\mathbf{B+1}} \tag{15}$$

III. Analysis and Method of Solution

The objective of the Milne problem is to determine the particle density everywhere in the medium and the angular distribution of the emergent intensity at the boundary x = 0, where the scattering law is described by Eq. (2). In a previous work [16-18] we introduced an efficient and accurate method for solving a class of half-space problems with isotropic scattering. In the present work the method is used to solve the Milne problem with synthetic scattering law. For this purpose equation (12) can be formally solved, subject to the boundary condition (14), to give

$$\boldsymbol{\Psi(x,\mu)} = \frac{\boldsymbol{\omega}}{\mathbf{2\mu}}\int_0^x \mathbf{e}^{-\frac{1}{\mu}(x-y)} \boldsymbol{\Phi}\mathbf{(y)dy} + \frac{\boldsymbol{\omega}\mathbf{R}}{\mathbf{2\mu}}\int_0^\infty \mathbf{e}^{-\frac{1}{\mu}(x+y)} \boldsymbol{\Phi}\mathbf{(y)dy}, \qquad \boldsymbol{\mu > 0} \tag{16}$$

and

$$\Psi\,(x,-\mu) = \frac{\omega}{2\mu}\int_x^\infty e^{-\frac{1}{\mu}(y-x)}\,\Phi(y)\,dy, \qquad \mu > 0 \tag{17}$$

Integrating Eqs. (16) and (17) over μ to obtain the following integral equation for $\Phi(x)$

$$\Phi(x) = \frac{\omega}{2}\int_0^\infty H(x,y)\,\Phi(y)\,dy, \tag{18}$$

with

$$H(x,y) = E_1\,(|x-y|) + RE_1\,(x+y), \tag{19}$$

Where E_n (x) denoting the exponential integral function of order n. Now, isolate the growing component of $\Phi(x)$ to rewrite Eq. (18) in the form

$$f(x) = S(x) + \frac{\omega}{2}\int_0^\infty H(x,)\,f(y)\,dy, \tag{20}$$

and the inhomogeneous term S(x) is defined by

$$S(x) = \frac{\omega\nu}{2}\left[e^{x/\nu}\,E_1\left(\frac{\nu+1}{\nu}x\right) - E_1(x)\right] + \frac{\omega\nu}{2}R\left[e^{-x/\nu}\,E_1\left(\frac{\nu-1}{\nu}x\right) - E_1(x)\right] \tag{21}$$

for $\omega < 1$, whereas for $\omega = 1$ one has

$$S(x) = \frac{1}{2}(1+R)\,E_3(x) \tag{22}$$

To completely determine the angular distributions $\Psi(x,\pm\mu)$ as given by Eqs. (16) and (17) we must solve Eq. (20) for f(x) and hence $\Phi(x)$. For this purpose, we introduce an appropriate parameters approximate expansion for f(x) and then determining the unknown expansion coefficients. A proper choice for the parametric expansion is of the form

$$f(x) = Ae^{-x/\nu} + \sum_{n=0}^{N} A_n\,E_{n+2}\,(x), \tag{23}$$

where A and A_n are expansion coefficients to be determined. Using this expansion, the angular distributions everywhere in the half-space can be calculated from, $\mu > 0$,

$$\Psi(x,\mu) = \frac{\omega\nu}{2}\left[\frac{e^{\frac{x}{\nu}} - e^{-\frac{x}{\mu}}}{\nu+\mu} + R\frac{e^{-\frac{x}{\mu}}}{\nu-\mu}\right] + \frac{\omega\nu}{2}\left[\frac{e^{-\frac{x}{\nu}} - e^{-\frac{x}{\mu}}}{\nu-\mu} + R\frac{e^{-\frac{x}{\mu}}}{\nu+\mu}\right]A$$
$$+ \frac{\omega}{2\mu}\sum_{n=0}^{N} A_n\,[F_{n+2}(x,\mu) + RF_{n+2}\,(\infty,-\mu)]e^{-\frac{x}{\mu}} \tag{24}$$

and

$$\Psi(x,-\mu) = \frac{\omega\nu}{2}\left[\frac{e^{\frac{x}{\nu}} - e^{-\frac{x}{\mu}}}{\nu-\mu} + R\frac{e^{-\frac{x}{\mu}}}{\nu+\mu}\right] +$$
$$\frac{\omega}{2\mu}\sum_{n=0}^{N} A_n\,[F_{n+2}(\infty,-\mu) - F_{n+2}\,(x,-\mu)]e^{\frac{x}{\mu}}, \tag{25}$$

where the functions F_n (x, μ) are as defined and explicitly given in Ref. [16]. Specializing equation (25) at x = 0 to get

$$\Psi(0,-\mu) = \frac{\omega\nu}{2}\left[\frac{1}{\nu-\mu} + \frac{A}{\nu+\mu}\right] + \frac{\omega}{2\mu}\sum_{n=0}^{N} A_n F_{n+2}\,(\infty,-\mu), \tag{26}$$

and from equation (9) the emergent angular takes the form

$$\boldsymbol{\psi(0,-\mu) = (1+R)\ \Psi(0,-\mu)} \qquad (27)$$

Another quantity, of physical interest, relevant to the Milne problem is the extrapolation length, z_0, which is the distance beyond the absorbing boundary at which the asymptotic part of the total flux extrapolates to zero. For example, in case of conservative scattering it is easy to show that

$$\frac{\mathbf{z_0}}{\mathbf{1+R}} = \frac{\mathbf{3}}{\mathbf{8}} + \frac{\mathbf{3}}{\mathbf{2}}\left[\frac{\mathbf{A}}{\mathbf{3}} + \sum_{\mathbf{n=0}}^{\mathbf{N}} \mathbf{A_n}\ \mathbf{J_{3,\,n+2}}\right], \qquad (28)$$

where J_{nm} are as defined and given in Ref. [16].

IV. The Expansion Coefficients

To solve for the expansion coefficients A and A_n, we substitute Eq. (23) into Eq. (20), then operate on the resulting expression first with the operator $\int_0^\infty e^{-x/\nu}.....dx,$ and second with $\int_0^\infty E_{m+2}(x).....dx,$ where m = 0, 1, 2, … , N. As a result of these two operations, one obtains (N+2) linear algebraic equations for (N+2) unknown expansion coefficients which are written as

$$\mathbf{AT}(\nu) + \sum_{\mathbf{n=0}}^{\mathbf{N}} \mathbf{A_n}\ \mathbf{T_n}(\nu) = \mathbf{d}(\nu), \qquad (29)$$

and

$$\mathbf{AT_m}(\nu) + \sum_{\mathbf{n=0}}^{\mathbf{N}} \mathbf{A_n}\ \mathbf{D_{mn}} = \mathbf{d_m}(\nu),\ \mathbf{m = 0,1,2,....,N} \qquad (30)$$

The analytic expressions for the integrals of T, T_n and D_{mn} are evaluated to give

$$\mathbf{T}(\nu) = \begin{cases} \dfrac{\nu}{\mathbf{2}}\left[\mathbf{1}-\boldsymbol{\omega}\nu(\mathbf{1+R})\ln\left(\dfrac{\nu+\mathbf{1}}{\nu}\right)-\boldsymbol{\omega}\mathbf{R}\dfrac{\nu}{\nu+\mathbf{1}}\right], & \mathbf{0<\boldsymbol{\omega}<1} \\ \dfrac{\mathbf{1}}{\mathbf{4}}(\mathbf{1-R}), & \boldsymbol{\omega}=\mathbf{1} \end{cases} \qquad (31)$$

$$\mathbf{T_n}(\nu) = \begin{cases} \dfrac{\boldsymbol{\omega}\nu}{\mathbf{2}}[\mathbf{Y^-_{n+2}} + \mathbf{RY^+_{n+2}} - (\mathbf{1+R})\ \mathbf{J_{1,n+2}}] & \mathbf{0<\boldsymbol{\omega}<1} \\ \dfrac{\mathbf{1}}{\mathbf{2}}(\mathbf{1-R})\ \mathbf{J}_{\mathbf{2},n+\mathbf{2}} & \boldsymbol{\omega}=\mathbf{1} \end{cases} \qquad (32)$$

and for D_{mn} one has

$$\mathbf{D_{mn}} = \mathbf{J_{m+2,n+2}} - \frac{\boldsymbol{\omega}}{\mathbf{2}}\mathbf{U_{m+2,\,n+2}} - \frac{\boldsymbol{\omega}}{\mathbf{2}}\mathbf{R\,I_{m+2,\,n+2}} \qquad (33)$$

where

$$(-\mathbf{1})^{\mathbf{m+n}}\ \mathbf{I_{mn}} = \tilde{\boldsymbol{\Psi}}_{\mathbf{m+n+1}} + \sum_{\mathbf{i=1}}^{\mathbf{n-1}}\frac{(-\mathbf{1})^{\mathbf{i}}}{\mathbf{i}}\boldsymbol{\Psi}_{\mathbf{m+n+1-i}} + \sum_{\mathbf{i=1}}^{\mathbf{m-1}}\frac{(-\mathbf{1})^{\mathbf{i}}}{\mathbf{i}}\boldsymbol{\Psi}_{\mathbf{m+n+1-i}} + \sum_{\mathbf{i=1}}^{\mathbf{m-1}}\sum_{\mathbf{j=1}}^{\mathbf{n-1}}(-\mathbf{1})^{\mathbf{i+j}}\frac{\mathbf{1}}{\mathbf{ij(m+n-i-j)}} \qquad (34)$$

In equations (33-35) $Y^{\pm}_m$, U_{mn}, $\tilde{\Psi}_m$ and Ψ_m are as given in Ref. (16). The inhomogeneous terms d and d_m, in the algebraic system of equations, are calculated to give

$$\mathbf{d}(\nu) = \begin{cases} \dfrac{\boldsymbol{\omega}\nu}{\mathbf{2}}\left[\dfrac{\nu}{\nu+\mathbf{1}} - \nu(\mathbf{1+R})\ \ln\left(\dfrac{\nu+\mathbf{1}}{\nu}\right) + \dfrac{\mathbf{R}}{\boldsymbol{\omega}}\right], & \mathbf{0<\boldsymbol{\omega}<1} \\ \dfrac{\mathbf{1}}{\mathbf{6}}(\mathbf{1+R}), & \boldsymbol{\omega} = \mathbf{1} \end{cases} \qquad (35)$$

and

$$\mathbf{d_m}(\nu)=\begin{cases}\dfrac{\boldsymbol{\omega}\,\nu}{\mathbf{2}}\left[\mathbf{Y}^{+}_{\mathbf{m+2}}+\mathbf{R}\ \mathbf{Y}^{-}_{\mathbf{m+2}}-(\mathbf{1}+\mathbf{R})\ \mathbf{J}_{\mathbf{1,m+2}}\right], & \mathbf{0}<\boldsymbol{\omega}<\mathbf{1}\\ \dfrac{\mathbf{1}}{\mathbf{2}}(\mathbf{1}+\mathbf{R})\ \mathbf{J}_{\mathbf{3,m+2}}\ , & \boldsymbol{\omega}=\mathbf{1}\end{cases} \tag{36}$$

V. Numerical Results

To test the present method, numerical results are given and compared with the available data. In Table 1 the convergence of the normalized emerging angular distribution, usually called law of darkening, for isotropic scattering at $\omega_0 = 0.5$ are listed and compared with those obtained by Chandrasekhar's method [19].

In Table 2 the extrapolation length of the Milne problem are listed and compared. Finally the law of darkening for $\omega_0 = 0.9$ and different values of ℓ, m and n are tabulated in Table 3. it is shown that the analytical expressions which we have obtained can be solved easily and our numerical results are in good agreement with other data.

Table 1. Convergence of the normalized exist angular intensity for isotropic scattering at $\omega_0 = 0.5$

N / μ	0	1	2	3	4	Ref. (19)
0.1	0296634	0.296484	0.296480	0.296480	0.296480	0.296480
0.2	0.344311	0.344292	0.344299	0.344299	0.344299	0.344299
0.3	0.401196	0.401219	0.401225	0.401225	0.401225	0.401225
0.4	0.473207	0.473244	0.473247	0.473248	0.473248	0.473248
0.5	0.569615	0.569656	0.569659	0.569659	0.569659	0.569659
0.6	0.707564	0.707608	0.707611	0.707611	0.707611	0.707611
0.7	0.923807	0.923855	0.923859	0.923859	0.923859	0.923859
0.8	1.315000	1.315060	1.315066	1.315066	1.315067	1.315067
0.9	2.245330	2.245424	2.245436	2.245436	2.245436	2.245436
1.0	7.360700	7.361006	7.361046	7.361046	7.361046	7.361046

Table 2. Convergence of the product $\omega_0 z_0$ in the case of isotropic scattering at two values of ω_0

ω_0 / N	0.5		0.8	
	This work	Ref. [10]	This work	Ref. [10]
0	0.72042464	0.72037813	0.71123671	0.71121121
1	0.72042496	0.72038065	0.71124365	0.71123343
2	0.72042497	0.72039667	0.71124368	0.71124205
3	0.72042497	0.72041055	0.71124368	0.71124363
4	0.72042497	0.72041835	0.71124368	0.71124371
5		0.72042223		0.71124369
6		0.72042396		0.71124368
7		0.72042468		0.71124368
8		0.72042502		0.71124368
9		0.72042496		0.71124368

Table 3. The exit angular intensity $\psi(0,-\mu)$, as calculated using the first order approximation, N=1, for $\omega_0 = 0.9$ and different indices (ℓ, m, n).

μ	$(\frac{2}{3},0,\frac{1}{3})$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$(\frac{1}{3},0,\frac{2}{3})$	$(0,\frac{1}{3},\frac{2}{3})$	$(0,\frac{2}{3},\frac{1}{3})$	$(0,0,1)$
0.1	0.131545	0.174768	0.194556	0.276978	0.256443	0.290988
0.2	0.167864	0.215765	0.235668	0.323347	0.299117	0.339962
0.3	0.207041	0.260146	0.278235	0.371526	0.342363	0.393636
0.4	0.251505	0.310769	0.324240	0.423911	0.388011	0.455718
0.5	0.304185	0.371036	0.375508	0.482662	0.437464	0.530567
0.6	0.369349	0.445899	0.434211	0.550338	0.492176	0.624539
0.7	0.453932	0.543413	0.503258	0.630363	0.553891	0.747991
0.8	0.570369	0.678019	0.586832	0.727668	0.624858	0.919532
0.9	0.743714	0.878830	0.691329	0.849797	0.708136	1.176704
1.0	1.033502	1.215034	0.827172	1.009051	0.808076	1.608779

VI. Conclusion

A method of solution based on using a suitable set of trial functions is presented for Milne problem in the case of extremely anisotropic scattering. The integro-differential equation is transformed into an equivalent fictitious one involving only isotropic scattering. The integral form is then solved based on trial functions. The results show that the convergence is rabid and lower order approximations are of sufficient accuracy for many situations. In fact, for pure scattering and weakly absorbing media, the zeros (or first) order approximation gives very good agreement with the available results.

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