Newtons Gravitational Field Equations for a Static Homogeneous Spherical Distribution of Mass in Rotational Spherical Polar Coordinates

Obagboye, L. F1; Howusu, S.X.K2

1 Theoretical Physics Programme, National Mathematical Centre Abuja P.M.B 118 Garki P.O.
2 Physics Department, Kogi State University Anyigba Kogi State, Nigeria.

Abstract: In this paper we formulate and solve Newton’s gravitational Field equations for a static homogeneous spherical distribution of mass in rotational spherical polar coordinates to pave the way for applications such as planetary theory in rotational spherical polar coordinates.

Key Words: Newton’s Gravitational Field Equations, Static Homogeneous Spherical distribution of mass, Rotational Spherical Polar Coordinates

Theory

It is well established that the Newton’s gravitational Field equations for the gravitational scalar potential $f$ due to a distribution of mass density $\rho$ is given by

$$\nabla^2 (x^\mu) = 4\pi G \rho_0 (x^\mu) \quad [1]$$

where $\nabla^2$ is the Euclidean Laplacian given by

$$\nabla^2 = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial u} \left[ \frac{h_2 h_3}{h_1} \frac{\partial}{\partial u} \right] + \frac{\partial}{\partial v} \left[ \frac{h_1 h_3}{h_2} \frac{\partial}{\partial v} \right] + \frac{\partial}{\partial w} \left[ \frac{h_1 h_2}{h_3} \frac{\partial}{\partial w} \right] \right\} \quad [2]$$

It follows immediately that

$$\nabla^2 = \frac{1}{u^2} \frac{\partial}{\partial u} \left( u^2 \frac{\partial}{\partial u} \right) + \frac{1}{v^2} \frac{\partial}{\partial v} \left( v^2 \frac{\partial}{\partial v} \right) + \frac{1}{w^2} \frac{\partial}{\partial w} \left( w^2 \frac{\partial}{\partial w} \right) \quad [3]$$

This reduces to

$$\nabla^2 = \frac{1}{u^2} \frac{\partial}{\partial u} \left( u^2 \frac{\partial}{\partial u} \right) + \frac{1}{v^2} \frac{\partial}{\partial v} \left( v^2 \frac{\partial}{\partial v} \right) + \frac{1}{w^2} \frac{\partial}{\partial w} \left( w^2 \frac{\partial}{\partial w} \right) \quad [4]$$

I. Research Elaborations

For a Static Homogeneous Spherical distribution of mass in Rotational spherical polar coordinates we consider two conditions following from equation (1)

$$\frac{1}{u^2} \frac{d}{du} \left( u^2 \frac{d}{du} \right) f(u) = 4\pi G \rho_0 ; u < R \quad [5]$$

$$\frac{1}{u^2} \frac{d}{du} \left( u^2 \frac{d}{du} \right) f(u) = 0; u > R \quad [6]$$

From equation (6)

$$\frac{d}{du} \left( u^2 \frac{d}{du} \right) f = A$$

$$\frac{d}{du} f = \frac{A}{u^2}$$

$$\Rightarrow f^+ = -\frac{A}{u} + B \quad [7]$$

From equation (5)

$$\frac{1}{u^2} \frac{d}{du} \left( u^2 \frac{d}{du} \right) f(u) = 4\pi G \rho_0$$

$$\frac{d}{du} \left( u^2 \frac{d}{du} \right) f(u) = \rho_0 G \rho_0 u^2$$

$$u^2 \frac{d}{du} f(u) = \frac{4}{3} \pi G \rho_0 u^2 + A$$
\[ f(u) = \frac{4}{3} \pi G \rho_0 \int u \, du \]
\[ f_p'(u) = \frac{2}{3} \pi G \rho_0 u^2 + B \]  

Solving for Complimentary \( f_c^- \)
\[ \nabla^2 f_c^- = 0 \]
\[ \frac{d}{du} \left( u^2 \frac{d}{du} f_c^- \right) = 0 \]

Setting \( f_c^- = DU^m \)  

It follows that \( m = -1 \) or 0
\[ \Rightarrow f_c^- = D \]
\[ f^- = f_p^+ + f_c^- \]
\[ = D + \frac{2}{3} \pi G \rho_0 u^2 \]

Knowing that \( f^+ = \frac{A}{U}; U > R \) and also that \( \rho_0 = \frac{3M_u}{4\pi R^3} \)

Therefore
\[ f^- = D + \frac{GM_0 u^2}{2R^3}; U < R \]  

\[ f^+ = \frac{A}{U}; U > R \]  

Now applying the conditions \( f^- = f^+; U = R \) and \( \frac{\partial f^-}{\partial u} = \frac{\partial f^+}{\partial u}; U = R \) we have
\[ \frac{\partial}{\partial u} \left[ D + \frac{GM_0 u^2}{2R^3} \right] = \frac{\partial}{\partial u} \left[ \frac{A}{U} + B \right] \]
\[ \Rightarrow A = -GM_0 \]

From \( f^- = f^+ \) we get
\[ D + \frac{GM_0 u^2}{2R^3} = \frac{A}{U} \]
\[ \therefore D = \frac{GM_0}{2R} \]  

The explicit Newton's gravitational Scalar potential field for a static homogeneous spherical distribution of mass is hence given by:
\[ f^+ = \frac{GM_0}{U} \]  

\[ f^- = \frac{GM_0}{2R} \left[ 1 + \frac{U^3}{R^2} \right] \]  

**II. Conclusion**

The Scalar potential field so derived can now be applied to derive the Riemannian gravitational intensity for the interior and exterior fields and hence decompose the Riemann's geodesic equation\(^3\) into the Riemann's acceleration part and a pure gravitational intensity or acceleration due to gravity. The scalar potential field plays a fundamental role in establishing a proof for the vanishing of the Riemannian curvature scalar in the Newton's gravitational field exterior to a static homogeneous spherical distribution of mass which has far reaching consequences in the theories of gravitation.

**References**