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Topological Quantum Control With Noisy Adiabatic Paths And Bloch Sphere Visualization

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Abstract:
Background: *Holonomic quantum computation leverages geometric phases to implement quantum gates, offering inherent robustness against certain types of errors. This makes it a promising candidate for fault-tolerant quantum computing. In this study, we explore adiabatic topological quantum control in a three-level Lambda system, focusing on the performance of holonomic gates under realistic, noisy conditions.*
Materials and Methods: *We employed numerical simulations to implement holonomic quantum gates via cyclic adiabatic evolutions in parameter space. Realistic imperfections were introduced through controlled Gaussian perturbations applied to the adiabatic trajectories. The time evolution of the quantum state was computed using small-time-step propagators derived from the system Hamiltonian. The resulting gates were projected onto the logical subspace of the qubit, and the evolution of the logical state was visualized on the Bloch sphere to assess gate performance.*
Results: *Despite the presence of Gaussian noise in the control parameters, the holonomic gates remained nearly unitary. This behavior underscores the robustness of topological quantum control. The Bloch sphere visualizations provided valuable insight into the dynamics of the logical qubit under noisy conditions.*
Conclusion: *Our results demonstrate the resilience of holonomic quantum gates to realistic perturbations, confirming the advantages of geometric control schemes for robust quantum computation. The combination of noisy trajectory modeling and Bloch sphere analysis offers a powerful approach for evaluating the fidelity and stability of quantum gates in practical implementations, with potential implications for fault-tolerant quantum computing architectures.*
Key Word*: Holonomic quantum control, topological quantum gates, adiabatic evolution, Bloch sphere visualization, noise robustness.*

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I. Introduction

Quantum control is essential for advancing quantum technologies, as it allows quantum states to be manipulated precisely for use in quantum computation, communication, and sensing[1]. It is crucial for improving these technologies. This is because it allows us to alter quantum states with great precision. These states can then be used in quantum computation, communication, and sensing. One approach that has attracted significant attention is topological quantum control, thanks to its ability to withstand certain types of error. These errors result from the geometric properties of quantum evolutions rather than from detailed dynamical parameters[2,3]. This strength makes topological quantum control a likely choice for fault-tolerant quantum computing structures[4].

Holonomic quantum computation, a subset of topological control, uses non-Abelian geometric phases, or holonomies, acquired by quantum systems undergoing cyclic adiabatic evolution in a parameter space[5,6]. Three-level Lambda (Λ) systems have been suggested as effective ways of implementing holonomic gates, where the logical qubit is encoded in a degenerate subspace that is resistant to certain control errors[7,8].

Despite their theoretical robustness, the practical implementation of holonomic gates is challenged by issues such as parameter fluctuations and environmental noise, which can reduce gate fidelity[9]. It is crucial to understand the effects of noise on the performance of topological quantum gates to realize scalable quantum processors[10].

In this study, we investigate adiabatic topological quantum control in a three-level Λ-type system with noisy parameter trajectories using numerical methods. We introduce Gaussian noise into the control angles, $\theta(t)$ and $\phi(t)$, to model realistic experimental imperfections, and analyze the impact of these imperfections on holonomic gate fidelity. To provide an intuitive visualization of qubit dynamics, we plot the evolution of the logical state on a Bloch sphere. This offers geometric insight into the control process.

The results demonstrate that holonomic gates can maintain a certain degree of unitarity and stability in the presence of significant noise, suggesting that geometric control schemes could be effective in noisy environments. Furthermore, the Bloch sphere visualization is a valuable tool for evaluating and refining quantum control protocols.

Theoretical framework

Topological quantum control exploits the geometric properties of quantum state evolution, offering robustness that is not reliant on detailed dynamics, but rather on the global features of the evolution path in parameter space[2]. Non-Abelian geometric phases (or holonomies) arising from cyclic adiabatic evolution within a degenerate space are utilized in holonomic quantum computation[5,3].

Three-Level lambda system Hamiltonian

A commonly studied system for holonomic control is the three-level Lambda (Λ) configuration, where two low-energy states $|0\rangle$ and $|1\rangle$ encode the logical qubit, and an excited state $|e\rangle$ serves as an ancillary level for control[7].

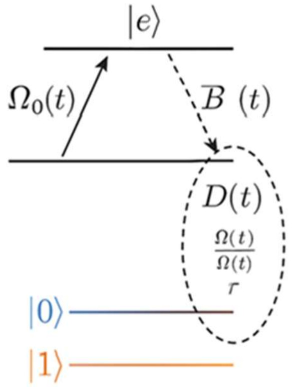

Figure 1: A three-level Lambda-type system used for adiabatic holonomic quantum computation.

The diagram shows a three-level arrangement where the two logical (ground) states $|0\rangle$ and $|1\rangle$ are joined to an excited state $|e\rangle$ by two control fields that change over time. These control fields are called $\Omega_0(t)$ and $\Omega_1(t)$.

As depicted in figure 1, these couplings define the so-called bright and dark states $|B(t)\rangle$ and $|D(t)\rangle$, which are coherent superpositions of the logical states. The dark state remains decoupled from the excited state throughout the evolution, enabling robust adiabatic control that is central to implementing holonomic quantum gates. The coupling scheme gives rise to two orthogonal instantaneous superposition states, and the dark state is given by:

$$|D(t)\rangle = \frac{\Omega_1(t)}{\Omega(t)}|0\rangle + \frac{\Omega_0(t)}{\Omega(t)}|1\rangle \quad (1)$$

where $\Omega(t) = \sqrt{|\Omega_0(t)|^2 + |\Omega_1(t)|^2}$ is the generalized Rabi frequency.

During adiabatic evolution, the system remains in the dark state subspace, avoiding population of the excited state $|e\rangle$. This mechanism enables robust geometric phase accumulation and forms the basis for non-Abelian holonomic quantum gates.

The system's Hamiltonian under resonant drives can be written as:

$$H(t) = \Omega_0(t)|0\rangle\langle e| + \Omega_1|1\rangle\langle e| + \Omega_0{}^*(t)|e\rangle\langle 0| + \Omega_1{}^*|e\rangle\langle 1| \quad (2)$$

where $\Omega_0(t)$ and $\Omega_1(t)$ are time-dependent Rabi frequencies controlling the transitions $|0\rangle \leftrightarrow |e\rangle$ and $|1\rangle \leftrightarrow |e\rangle$, respectively.

Parameterizing these couplings as:

$$\Omega_0(t) = \Omega\cos\theta(t),\ \Omega_1(t) = \Omega\sin\theta(t)e^{i\phi(t)} \quad (3)$$

with Ω representing a constant amplitude, control is achieved through the gradual variation of angles.$\theta(t)$ and $\phi(t)$ over a closed path in the (θ, ϕ) parameter space.

Adiabatic evolution and holonomies

If the evolution is adiabatic and cyclic, the system remains within the degenerate subspace spanned by the dark states, which do not populate the excited level $|e\rangle$ [6]. The change in the logical qubit space is like a turning around of the path, which is determined by the shape of the path:

$$U = \mathcal{P}\exp\left(-\oint_{C}\mathcal{A}\right) \quad (4)$$

where $\mathcal{P}$ denotes path ordering, C is the closed loop in parameter space, and $\mathcal{A}$ is the non-Abelian Berry connection[5].

Noise in parameter trajectories

Realistic implementations involve noise and control imperfections that cause deviations from ideal paths and potentially affect gate fidelity [9]. To model these effects, we introduce Gaussian noise perturbations, $\delta\theta(t)$ and $\delta\phi(t)$, which are added to the nominal control angles to produce noisy trajectories.

$$\theta'(t) = \theta(t) + \delta\theta(t),\ \phi'(t) = \phi(t) + \delta\phi(t) \quad (5)$$

Studying the system's response to these noisy trajectories enables assessment of the robustness of holonomic gates under realistic conditions.

Projection to logical subspace and Bloch sphere representation

Although the system is three-level, quantum information is encoded in the two-dimensional logical subspace span $\{|0\rangle, |1\rangle\}$. After time evolution under the noisy Hamiltonian, the final state is projected onto this logical subspace. The logical qubit state $|\psi_L\rangle$ can be represented on the Bloch sphere by mapping the two-level state vector to a three-dimensional vector r $= (x, y, z)$, where:

$$x = 2Re(\alpha^*\beta), y = 2Im(\alpha^*\beta), z = |\alpha|^2 - |\beta|^2 \quad (6)$$

II. Material And Methods

To investigate the robustness of adiabatic topological quantum control under realistic noise conditions, we numerically solve the time-dependent Schrödinger equation for a three-level Lambda system with a time-dependent Hamiltonian H(t). The main steps of the numerical procedure are described below.

Generation of noisy trajectories

The ideal control parameters $\theta(t)$ and $\phi(t)$ define a smooth, closed path in the parameter space, commonly parametrized as[7]:

$$\theta(t) = \frac{\pi}{2}\left(1 - \cos\left(\frac{2\pi t}{T}\right)\right), \phi(t) = \frac{2\pi t}{T} \quad (7)$$

where T is the total evolution time. To simulate imperfections, present in realistic experimental settings, we add independent Gaussian noise to both parameters at discrete time intervals[9]:

$$\theta'(t_i) = \theta(t_i) + \delta\theta_i, \phi'(t_i) = \phi(t_i) + \delta\phi_i \quad (8)$$

with $\delta\theta_i$, $\delta\phi_i \sim N(0, \sigma^2)$, where σ quantifies noise strength.

Construction of time evolution operators

The time evolution over a small-time step Δt is approximated by the unitary propagator.

$$U(t_i + \Delta t, t_i) = e^{(-iH(t_i)\Delta t)} \quad (9)$$

where $H(t_i)$ is evaluated at the noisy parameters $\theta'(t_i), \phi'(t_i)$[14]. The full evolution operator is obtained as the time-ordered product of these propagators:

$$U(T, 0) = \sum_{i=0}^{N-1} U(t_{i+1}, t_i) \quad (10)$$

with $N = \frac{T}{\Delta t}$.

Projection onto the logical qubit subspace

The first logical qubit state, $|\psi_L(0)\rangle$, is found within the two-dimensional dark space of the three-level system. After evolving under $U(T, 0)$, the resulting state, $|\psi(T)\rangle$, may contain some population outside the logical subspace due to noise-induced leakage[6]. We can analyze gate performance by projecting the final state back onto the logical space spanned by $|0\rangle$ and $|1\rangle$.

Visualization of the geometric phase as an area in $(\boldsymbol{\theta}, \boldsymbol{\phi})$ space

The representation of the geometric phase as a surface area in the angular parameter space—characterized by the polar angle θ and azimuthal angle ϕ —offers a robust and insightful geometric interpretation of phase accumulation in quantum systems. Trajectories on the Bloch sphere are often used to illustrate this visualization. The geometric phase is the solid angle subtended by a closed path in this context.

Figure 2 illustrates the trajectory of a quantum system in the control parameter space (θ, ϕ) and its relation to the geometric phase integral. The lower sinusoidal blue curve corresponds to the polar angle θ as a function of the azimuthal parameter ϕ, tracing a closed loop in parameter space. The upper dark blue curve represents $cos^2(\theta)$, whose integral over ϕ determines the geometric phase $\gamma = \oint cos^2(\theta) d\phi$. The light blue shaded area under the $cos^2(\theta)$ curve illustrates the accumulated geometric phase, emphasizing its topological nature and resilience to local perturbations. A red dot marks the initial point of the evolution, highlighting the cyclic character of the trajectory. This dual representation links the system's trajectory in (θ, ϕ) space to the phase acquired during adiabatic holonomic evolution.

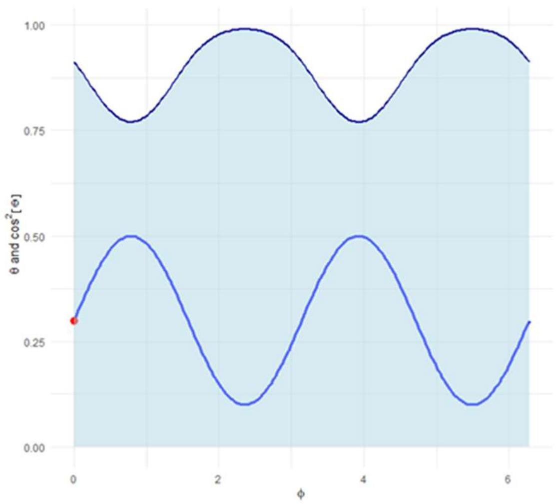

Figure 2. Trajectory in parameter space and geometric phase integral.

In mathematics, the geometric Berry phase obtained during a periodic, adiabatic process can be described as a surface integral of the Berry curvature across the area enclosed by the trajectory in parameter space. This makes a clear connection between the geometric phase and the solid angle, showing how the ideas of quantum mechanics can be linked to real-world geometry[11].

Not only is conceptual clarity enriched by this geometric approach, but it also serves as a practical analytical tool in the study of two-level quantum systems, such as spin-½ particles, qubits, and polarized light setups. It allows for a straightforward and visual comprehension of phase accumulation, circumventing the necessity for complex algebraic manipulations[12].

From an applied perspective, picturing the geometric phase as a tool can aid in the planning, management, and review of trials across areas such as quantum information processing, topological materials, and precision interferometry. The geometric representation of the phase offers an intuitive method for manipulating quantum phases. This is a foundational aspect for implementing quantum gates. It is also important for detecting topological invariants, and it helps to optimize interferometric sensitivity[13].

Bloch sphere visualization

The projected logical qubit state $|\psi_L(T)\rangle = \alpha|0\rangle + \beta|1\rangle$ is mapped onto the coordinates of the Bloch sphere[6], which allows for an intuitive geometric characterized of qubit evolution and gate fidelity [10,14].

The Bloch sphere is a fundamental tool in quantum mechanics, particularly for studying and visualizing the quantum states of two-level systems, such as qubits. It shows quantum states as points on a sphere. Each point on the sphere shows a different pure state of the qubit. This representation makes it easier for us to understand complex quantum phenomena such as superposition, entanglement and quantum gates, which are all fundamental to quantum computing and quantum information theory.

In quantum computing, where qubits are the basic units of information, the Bloch sphere assists with designing and analyzing quantum algorithms. The visual model helps researchers and engineers to conceptualize quantum entanglement and interference, which are pivotal for tasks such as quantum error correction and quantum cryptography [15].

In sum, the Bloch sphere serves to advance the didactic comprehension of quantum mechanics and quantum information, thus rendering it an indispensable instrument for theoretical study and practical applications in the domain of quantum technologies.

Topological quantum control with noisy adiabatic paths: A detailed example

We provide a detailed example illustrating how topological features in quantum control can remain robust under noisy adiabatic evolution, using a single qubit system subjected to a rotating magnetic field. This setup is well-known for exhibiting geometric Berry phases and serves as a canonical example for visualizing topological control on the Bloch sphere [11,16,17].

System description: Qubit in a rotating magnetic field

We consider a two-level system (qubit) governed by a time-dependent Hamiltonian:

$$H(t) = \frac{\hbar}{2}\vec{\Omega}(t) \cdot \vec{\sigma} \quad (11)$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices, and $\vec{\Omega}(t)$ represents an effective magnetic field of fixed magnitude Ω whose direction varies slowly over time. Specifically, the control path in parameter space is defined by:

$$\vec{\Omega}(t) = \Omega \begin{pmatrix} sin\theta(t)cos\phi(t) \\ sin\theta(t)sin\phi(t) \\ cos\theta(t) \end{pmatrix} \quad (12)$$

In the noiseless case, $\theta(t) = \theta_0$ is constant and $\phi(t)$ varies slowly from 0 to 2π, sweeping out a closed loop on the Bloch sphere. This adiabatic evolution leads to the accumulation of a Berry phase, which is purely geometric and given by:

$$\gamma = -\pi(1 - cos\theta_0) \quad (13)$$

corresponding to the solid angle enclosed by the loop on sphere [11].

Noisy control path

We now introduce noise into the adiabatic path by allowing small random perturbations in the polar angle $\theta(t)$. Specifically, we consider:

$$\theta(t) = \theta_0 + \delta\theta(t) \quad (14)$$

where $\delta\theta(t)$ is a zero-mean Gaussian stochastic process with correlation function:

$$\langle \delta\theta(t)\delta\theta(s) \rangle = \sigma^2 e^{\frac{-|t-s|}{\tau_c}} \quad (15)$$

with noise strength σ and correlation time τ_c. This leads to stochastic variation in the geometric phase:

$$\tilde{\gamma} = -\pi\big(1 - cos(\theta_0 + \delta\theta(t))\big) \quad (16)$$

Expanding to second order:

$$\tilde{\gamma} \approx -\pi\left(1 - cos\theta_0 + \frac{1}{2}\sigma^2 cos\theta_0\right) \quad (17)$$

Taking the expectation over the noise ensemble, we obtain the mean and variance of the noisy geometric phase:

$$Var(\tilde{\gamma}) = \pi^2 sin^2\theta_0 \cdot \sigma^2 \quad (18)$$

These expressions indicate that the geometric phase is robust to first-order fluctuations in the control path, but second-order contributions lead to dephasing and decoherence [18].

Bloch sphere visualization

To visualize the effect of noise on the system's trajectory, we numerically simulate the qubit's evolution using the stochastic Schrödinger equation:

$$\frac{d}{dt}\vec{r}(t) = \left[\vec{\Omega}(t) + \delta\vec{\Omega}(t)\right] \times \vec{r}(t) \quad (19)$$

where $\vec{r}(t)$ is the Bloch vector and $\delta\vec{\Omega}(t)$ is the noise-induced fluctuation in the control field. The result is a trajectory that deviates from the ideal conical path, leading to a "blurring" of the endpoint on the Bloch sphere and reduction of visibility in interference experiments [19].

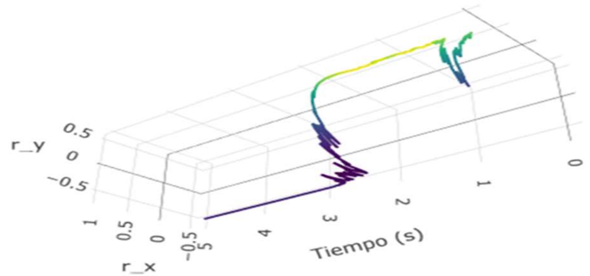

Figure 3. Time evolution of a qubit subjected to a rotating magnetic field.

The trajectory in the (r_x, r_y) plane, shown as a function of time, illustrates how the system's topological features remain robust under noisy adiabatic evolution. This setup, known for exhibiting geometric Berry phases, serves as a canonical example for visualizing topological control on the Bloch sphere.

The evolution of the Bloch vector is governed by both a deterministic precession and stochastic noise. The deterministic part comes from the system's Hamiltonian, which leads to precession of the Bloch vector around the z-axis at a constant frequency Ω_0. This precession is driven by a frequency of 1 Hz, as described by $\Omega_0 = [0,0,2\pi]$, which causes the vector to rotate around the z-axis, tracing a circular path on the Bloch sphere's xy-plane[15,20]. The disturbance is the second element, and it is represented as a white Gaussian disturbance. This is included at each time interval. This noise term introduces fluctuations into the evolution of the vector, causing random deviations from the ideal precession.

The vector's time evolution is approximated using the Euler–Maruyama method, which employs small time steps Δt to simulate the dynamics. At each stage, the Bloch vector is revised based on the noise term and the deterministic precession, making sure that it stays to prevent numerical errors[15, 19]. The noise term is scaled by $\sqrt{\frac{2D}{\Delta t}}$, where D represents the noise intensity, adding random perturbations to each component of the vector.

The movement of the Bloch vector is shown in three dimensions. The time dimension is shown on the x-axis, the r_x and r_y components are shown along the y and z axes. The hue of the line in the three-dimensional representation is associated with the r_z component, thereby emphasizing the manner in which the qubit's state vacillates along the vertical axis due to the presence of noise. The trajectory of the Bloch vector perpetually fluctuates, exhibiting a combination of regular and random movements.

In the absence of noise, the vector would follow a smooth, circular path in the xy-plane, rotating around the z-axis. However, the introduction of stochastic noise causes the path to deviate, resulting in irregular oscillations in all three components of vector[11,21]. These fluctuations are most evident in the r_y and r_z components, which exhibit erratic behavior as the noise perturbs the state. Despite this, the overall evolution retains a periodic character, reflecting the underlying deterministic precession around the z-axis[4].

The evolution of the Bloch vector under the influence of both deterministic precession and stochastic noise provides a comprehensive understanding of qubit dynamics in realistic conditions. The visualization demonstrates how noise affects the quantum state, leading to irregular fluctuations while preserving the periodic precession. This analysis is essential for understanding the behavior of quantum systems in noisy environments and is relevant for quantum computing and information processing, where noise can significantly impact the reliability of quantum operations[2].

Topological robustness and implications

Despite the presence of noise, the topological nature of the geometric phase imparts robustness to the control protocol. The key feature is that the phase depends on the area enclosed in parameter space, not the details of how the path is traversed. For systems that undergo cyclic adiabatic evolution, small deviations in the path shape lead to small second-order corrections, preserving the qualitative behavior of evolution[2,20].

This property is of practical importance in quantum computing, particularly in holonomic and geometric quantum gates, where control fidelity benefits from topological protection against noise[6].

III. Result

We performed numerical simulations of the adiabatic holonomic quantum gate in a three-level Lambda system subjected to Gaussian noise in the control parameters $\theta(t)$ and $\phi(t)$. The simulations were conducted for varying noise amplitudes σ and total evolution time T, with the goal of assessing the robustness of the topological gates and visualizing the qubit dynamics on the Bloch sphere.

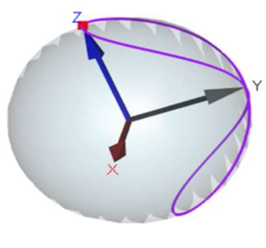

Figure 4. This figure illustrates the numerical simulation of the qubit's state evolution on the Bloch sphere under the adiabatic variation of control parameters.

The Bloch sphere representation illustrates the state of a two-level quantum system, with the red dot marking the initial state. The purple trajectory depicts the closed path traced by the state vector as the system undergoes adiabatic evolution driven by a continuous and cyclic variation of the control parameters.

This closed loop is a hallmark of a geometric phase (specifically, the Berry phase) acquired by the quantum state over the course of the evolution. The geometric phase depends solely on the trajectory's geometry on the Bloch sphere. It does not depend on the specific details of the temporal evolution. This includes the rate of traversal. As a result, it is inherently robust to certain types of control imperfections and noise, making it a powerful feature for fault-tolerant quantum computation and quantum control protocols.

The coordinate axes (X, Y, Z) define the orientation of the Bloch sphere, with the state vector initially aligned near the Z-axis. Upon completion of the closed loop, although the state returns to its initial position in Hilbert space, it has undergone a global phase transformation. This topological effect underlies the interest in geometric quantum gates for building resilient quantum technologies.

Noise effects on state evolution

The evolution of a logical qubit state on the Bloch sphere under both ideal (noise-free) and noisy conditions is presented in figure 5, with the trajectory traced by the qubit during a cyclic holonomic quantum gate operation being illustrated. The blue line shows the perfect, noise-free situation, while the green, orange, and red lines show noisy changes with random noise added (with $\sigma = 0.01, 0.05,$ and 0.1 radians, respectively).

In the best case, the qubit moves in a straight line on the Bloch sphere, going back to its starting point. This behavior is consistent with high-fidelity holonomic operations, in which the cyclic evolution ensures robustness against certain control imperfections and dynamical noise — a key theoretical advantage of holonomic quantum computation[6,7,22].

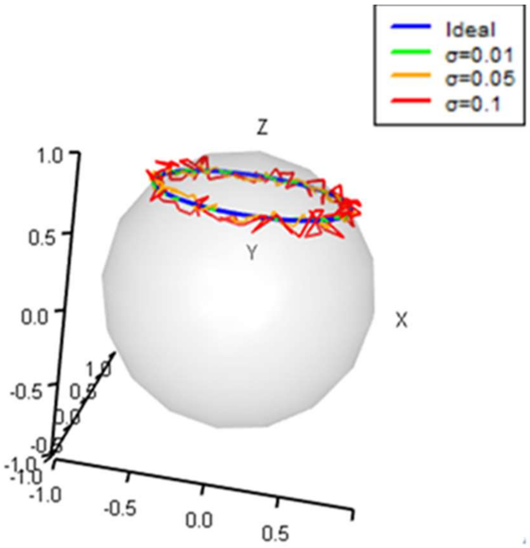

Figure 5. Bloch Sphere trajectories of a logical qubit under varying noise levels.

As noise is introduced, the trajectories increasingly deviate from the ideal path. Nevertheless, even when there is only moderate noise ($\sigma = 0.05$), the final state stays close to the target point. This demonstrates the high resistance of holonomic gates to changes in parameters[9]. This resilience arises from the geometric nature of holonomic operations, which depend only on the global properties of the control path. Nevertheless, larger noise amplitudes (e.g., $\sigma = 0.1$) cause more pronounced deviations and result in visible distortions and fluctuations. These lead to partial leakage out of the logical subspace; a phenomenon reflected in the reduced purity of the projected qubit state. Such degradation underscores the limitations of passive robustness and highlights the necessity of combining geometric design with error mitigation techniques for practical quantum gate implementations.

Characterization of noisy qubit dynamics on the Bloch sphere and associated geometric phases.

The interactive control of noise amplitude and total evolution time provides insight into their effects on qubit state evolution and geometric phase accumulation.

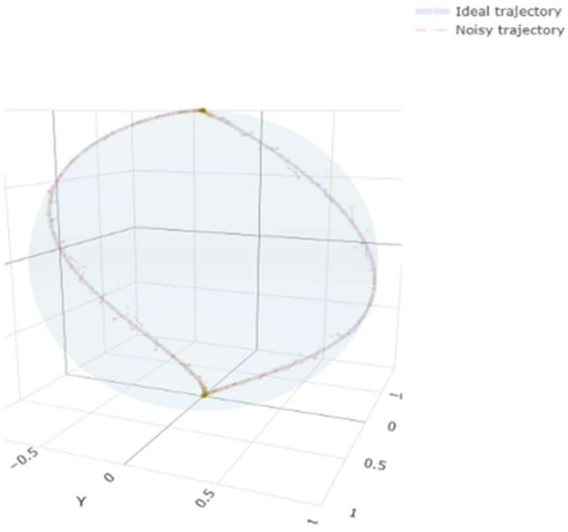

Figure 6. The Bloch sphere shows qubit state evolution under an adiabatic holonomic quantum gate.

The blue curve represents the ideal, noise-free path, whereas the red dashed curve illustrates the noisy trajectory with perturbations ($\sigma = 0.05$ rad). Deviations in phase are introduced by noise, which reduces gate fidelity and highlights the trade-off between robustness and noise sensitivity in topological quantum control.

This figure shows qubit state evolution on the Bloch sphere under a cyclic adiabatic control protocol. The blue curve represents the ideal (noise-free) trajectory, following a smooth closed path corresponding to the intended holonomic gate operation, which accumulates a well-defined Berry phase. The red dashed curve illustrates the turbulent trajectory, where arbitrary perturbations with a magnitude of σ impact the control parameters θ and ϕ, resulting in variations that modify the route.

Two key effects are caused by noise-induced deviations: first, the accumulated geometric phase along the noisy path differs from the ideal Berry phase, with phase errors in the quantum gate being reflected; second, gate fidelity is reduced by noisy evolution by shifting the state from the target path, with susceptibility to decoherence and leakage being potentially increased. These effects highlight the trade-off in topological quantum control: while geometric phases offer intrinsic robustness due to their dependence on global properties, local noise can degrade gate performance by altering the path's geometry. So, what we're saying here is that keeping an eye on noise amplitude and evolution time is important for getting the right balance between adiabaticity and noise resilience in practical quantum systems.

Evolution of the polar angle θ as a function of the azimuthal angle ϕ for a single noisy trajectory

The temporal evolution of the polar angle, θ, as a function of the azimuthal angle, ϕ, for an individual trajectory subject to stochastic perturbations characterized by Gaussian noise is illustrated in figure 7. This representation clearly shows how random fluctuations influence the spherical coordinates that define the qubit's state vector on the Bloch sphere.

The underlying smooth sinusoidal profile corresponds to the deterministic, noise-free trajectory predicted by the system's coherent evolution. Irregular deviations arising from angular perturbations introduced by the noise process have been superimposed on this baseline, effectively modelling angular diffusion within the qubit's Hilbert space.

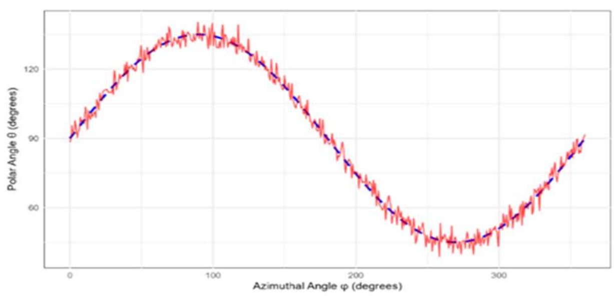

Figure 7. Trajectory of noise in (θ, φ) coordinates.

This plot serves as a critical diagnostic tool for quantitatively characterizing the angular diffusion mechanisms responsible for decoherence phenomena. Specifically, it demonstrates that, for moderate noise amplitudes, stochastic fluctuations primarily induce distortions to the trajectory without completely obliterating its inherent periodicity. As a result, the qubit's development has a clear repeating pattern, even when there is noise.

These observations complement the comprehensive three-dimensional visualization presented in figure 5, offering a more granular perspective on the interplay between coherent dynamics and noise-induced angular diffusion. Together, these analyses deepen our understanding of how Gaussian fluctuations modulate the qubit's trajectory on the Bloch sphere, informing theoretical models and experimental approaches aimed at mitigating decoherence effects in quantum information processing.

Pure holonomic evolution: A smooth cyclic path

Figure 8 shows a single ideal trajectory of a logical qubit evolving smoothly on the Bloch sphere under a holonomic quantum gate. A complex cyclic path that does not retrace itself but eventually returns close to the initial state is illustrated by the figure. The trajectory features nontrivial winding along both azimuthal and polar directions, suggesting a non-Abelian geometric phase accumulation.

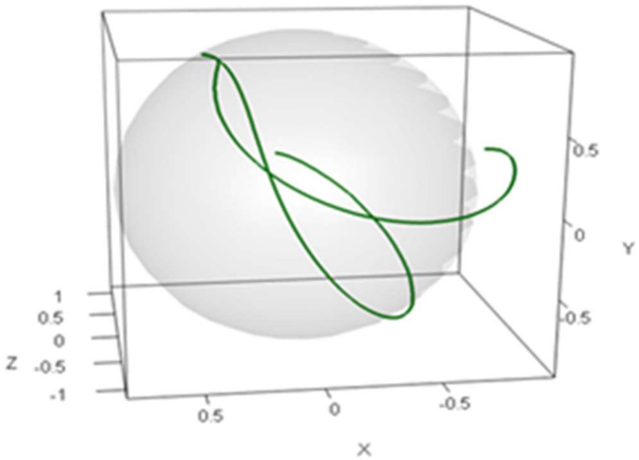

Figure 8. The evolution of the state in Bloch's sphere under ideal conditions.

This smooth evolution provides a baseline for comparison with noisy counterparts, which is useful for determining the impact of noise on the system. It emphasizes the power of geometric control and the cyclic nature of holonomic operations, both of which are key to fault-tolerant gate design.

Together, these three visualizations illustrate the resilience and vulnerability of holonomic gates in quantum computation. High-fidelity performance under weak noise is ensured by the intrinsic geometric protection, but stronger fluctuations can ultimately limit this by inducing trajectory deviation and space leakage.

Future work should explore real-time feedback and error-correcting protocols adapted to geometric control schemes, as well as hybrid approaches that combine holonomic gates with dynamical decoupling or reservoir engineering.

Fidelity analysis

To quantitatively evaluate gate performance, we computed the average fidelity F between the ideal and noisy final states over multiple noise realizations, defined as

$$F = \left|\langle\psi_{ideal}|\psi_{noisy}\rangle\right|^2 \tag{20}$$

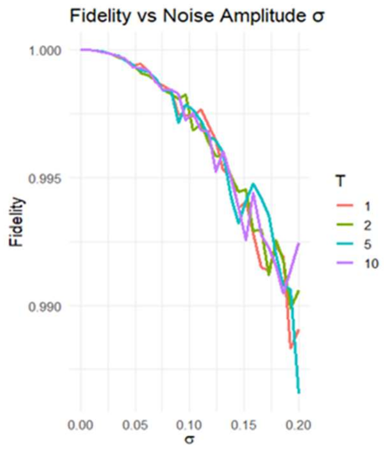

Figure 9. Fidelity as a function of noise amplitude σ for various total evolution times T.

Fidelity is enhanced under weak noise by longer adiabatic durations, which better satisfy the adiabatic condition[8]. However, extended exposure to stochastic perturbations eventually reduces fidelity, indicating a trade-off between adiabatic accuracy and noise sensitivity[10,23].

In the low-noise regime (i.e., for small values of σ), the figure clearly shows that increasing the total evolution time T improves fidelity. This behavior can be understood in light of the adiabatic theorem, which states that a quantum system will remain in its instantaneous eigenstate provided that the Hamiltonian varies sufficiently slowly and that the spectral gap remains open. Longer evolution times make it easier to meet the adiabatic conditions, as shown by the adiabatic criterion[8], which makes sure the ground state can be tracked more accurately and with more precision. The figure supports this interpretation by showing that, for small noise amplitudes, curves corresponding to larger T values consistently lie above those corresponding to shorter evolution durations.

However, this trend will not persist indefinitely. As the noise amplitude increases, the benefit of a longer evolution time decreases and can even become negative. Specifically, for moderate to large values of σ, the fidelity decreases more steeply with longer T due to cumulative decoherence and stochastic perturbations. An increased exposure duration inherently permits greater noise amassing, which degrades quantum coherence and eventually disturbs the adiabatic path. Rather than remaining confined to the desired instantaneous eigenstate, the system undergoes an increasing number of transitions to undesired states due to noise influence.

This behavior illustrates a fundamental trade-off between adiabaticity and noise resilience, which are both important considerations in this field. Slower evolution ensures better adherence to the ideal adiabatic path; however, it also results in prolonged exposure to environmental noise, which acts destructively on the coherence and fidelity of the quantum state. As Sjöqvist explains[10], these trade-offs are key to the design of strong quantum control protocols, especially in open systems or experimental settings where it is impossible to completely get rid of noise.

So, as you can see in figure 9, there's this really important point where the benefits of adiabaticity meet the negative effects of noise. For any given noise amplitude, neither the shortest nor the longest evolution time is universally optimal. Instead, system designers must carefully select T based on the noise characteristics in question and the desired fidelity threshold. This has profound implications for quantum computation and simulation strategies, where maximizing fidelity under realistic constraints is essential.

The simulations also tracked population leakage into the excited auxiliary level, $|e\rangle$. The results show that leakage remains minimal for small noise levels, but increases with σ, which is consistent with previous theoretical studies[9]. Unitarity is lost and gate infidelity is caused by this leakage, highlighting the importance of error mitigation strategies in practical implementations.

The numerical results confirm that holonomic quantum gates implemented via adiabatic parameter cycles exhibit resilience to moderate noise, maintaining high fidelity and coherence in the logical qubit subspace. Visualization on the Bloch sphere provides a clear geometric understanding of the dynamics and aids in identifying error sources. These findings support the viability of holonomic quantum control for fault-tolerant quantum computing architectures under realistic experimental conditions.

IV. Discussion

The numerical simulations presented in this study shed light on the robustness and limitations of adiabatic holonomic quantum control in the presence of noise. It is shown by our results that, consistent with previous theoretical predictions, intrinsic resilience to parameter fluctuations is exhibited by holonomic gates due to their geometric nature. This property is very good for quantum information processing, where control problems and noise from the environment are big problems.

Only slight deviations from the ideal cyclic trajectories are caused by moderate noise, as revealed by the Bloch sphere visualization. This preserves the coherence and relative phase of the logical qubit states. This corroborates the earlier findings that the geometric phases underlying holonomic gates are stable against minor perturbations in the control parameters[6,14]. Nonetheless, as the amplitude of the noise increases, there is a concomitant increase in leakage into the excited ancillary state, which results in degradation of gate fidelity and indicates a practical limit to noise tolerance.

The fidelity analysis reveals a trade-off between the adiabaticity condition and exposure to noise. While longer evolution times better satisfy adiabatic requirements and reduce non-adiabatic errors, they also increase the time during which noise can accumulate. Therefore, optimizing gate duration is crucial for maximizing performance in noisy.

From a practical point of view, these results support ongoing work to use holonomic quantum gates in platforms such as trapped ions, superconducting circuits and semiconductor quantum dots. In these systems, it is possible to precisely control the way the system changes over time. Future work could explore integrating error-correcting codes and dynamical decoupling techniques with holonomic control to further suppress the effects of noise and leakage.

In summary, the robustness of topological quantum control demonstrated here reinforces its potential as a building block for scalable, fault-tolerant quantum computation. The amalgamation of numerical simulations and geometric visualization tools furnishes a robust framework for the design, optimization and benchmarking of holonomic quantum gates under realistic operational conditions.

V. Conclusion

In this study, we numerically investigated the implementation of adiabatic holonomic quantum control in a three-level Lambda system, with a particular focus on the impact of Gaussian noise on the control parameters. Our simulations show that holonomic gates are robust against moderate noise, preserving the coherence and fidelity of logical qubit states. The geometric character of the control protocol underlies this durability, as evidenced by the consistent paths seen on the Bloch sphere.

However, noise-induced deviations and leakage to the ancillary excited state impose practical limits on gate fidelity, particularly at higher noise levels and longer gate durations. This highlights the importance of optimizing the adiabatic evolution time and incorporating noise mitigation techniques in experimental realizations, which are essential for achieving accurate results and ensuring the reliability of the findings.

Our findings lend support to the notion that topological quantum control has the potential to serve as a promising approach for fault-tolerant quantum computation, seamlessly integrating geometric robustness with experimental feasibility. Future studies integrating error correction and exploring non-adiabatic holonomic schemes could further improve the performance and applicability of these protocols.

Appendix A:

Let $H(t)$ be a time-dependent Hamiltonian.

$$H(t) = \frac{\hbar}{2}\vec{\Omega}(t)\cdot\vec{\sigma} \tag{21}$$

Where $\vec{\Omega}(t)$ describes a magnetic field of constant magnitude that slowly rotates in space. Consider a qubit in a rotating magnetic field, with

$$\vec{\Omega}(t) = \Omega(\sin\theta\cos\phi(t), \sin\theta\sin\phi(t), \cos\theta) \tag{22}$$

where

Ω: field strength (constant),

θ: fixed angle with respect to the zzz-axis,

$\phi(t) = \omega t$: phase that rotates with a slow angular frequency ω.

This field traces out a cone in parameter space → a closed path on the Bloch sphere if ϕ goes from $0 \to 2\pi$.

The Hamiltonian becomes.

$$H(t) = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi(t)} \\ \sin\theta e^{i\phi(t)} & -\cos\theta \end{pmatrix} \tag{23}$$

The eigenvalues are:

$$E_{\pm} = \frac{\hbar\Omega}{2} \tag{24}$$

The eigenvector associated with E_+ (instantaneous state) is:

$$|+(\theta,\phi(t))\rangle = \begin{pmatrix} cos\left(\frac{\theta}{2}\right) \\ sin\left(\frac{\theta}{2}\right)e^{i\phi(t)} \end{pmatrix} \quad (25)$$

The Berry phase for this state, if $\phi(t)$ completes a full cycle from $0 \to 2\pi$, is

$$\gamma_+ = i\oint \langle + |\nabla_{\vec{\lambda}}| + \rangle \cdot d\vec{\lambda} \quad (26)$$

Since $\vec{\lambda}(\theta,\phi)$ and θ is constant, only ϕ varies.
We compute:

$$\frac{d}{dt}|+\rangle = \frac{d\phi}{dt}\frac{\partial}{\partial\phi}|+\rangle = \dot{\phi}\frac{\partial}{\partial\phi}|+\rangle \quad (27)$$

Thus:

$$\gamma_+ = i\int_0^T \langle + \left|\frac{d}{dt}\right| + \rangle\, dt = i\int_0^T \dot{\phi}\,\langle + \left|\frac{\partial}{\partial\phi}\right| + \rangle\, dt = i\int_0^{2\pi} \langle + \left|\frac{\partial}{\partial\phi}\right| + \rangle\, d\phi \quad (28)$$

Computing:

$$\frac{\partial}{\partial\phi}|+\rangle = \begin{pmatrix} 0 \\ isin\left(\frac{\theta}{2}\right)e^{i\phi} \end{pmatrix}, \langle +| = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & sin\left(\frac{\theta}{2}\right)e^{i\phi} \end{pmatrix} \quad (29)$$

Therefore:

$$\langle + \left|\frac{\partial}{\partial\phi}\right| + \rangle = isin\left(\frac{\theta}{2}\right) \quad (30)$$

Finally:

$$\gamma_+ = i\int_0^{2\pi} isin\left(\frac{\theta}{2}\right) d\phi = -\pi(1 - cos\theta) \quad (31)$$

This is exactly the solid angle subtended by the magnetic field path on the Bloch sphere. The geometric phase does not depend on Ω nor on the rotation speed (ω), if the evolution is adiabatic.
Now suppose $\theta \to \theta + \delta\theta(t)$, with $\delta\theta(t)$ small. The phase becomes.

$$\gamma_+ = -\pi\big(1 - cos(\theta + \delta\theta(t))\big) \quad (32)$$

By Taylor expansion:

$$\tilde{\gamma}_+ \approx -\pi\left(1 - cos\theta + \delta\theta sin\theta + \frac{1}{2}(\delta\theta)^2 cos\theta\right) \quad (33)$$

Averaging over Gaussian noise with $\langle\delta\theta\rangle = 0$, we have:

$$\tilde{\gamma}_+ \approx -\pi\left(1 - cos\theta + \frac{1}{2}\langle(\delta\theta)^2\rangle cos\theta\right) \quad (34)$$

And the variance of the phase is:

$$Var(\tilde{\gamma}_+) = \pi^2 sin^2\theta \cdot \langle(\delta\theta)^2\rangle \quad (35)$$

The geometric phase is therefore not perfectly robust against stochastic fluctuations.

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